

# Lossy Compression with a Short Processing Block: Asymptotic Analysis

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**Abstract**—Finite-blocklength analysis of lossy compression can have a variety of different motivations. First, the source sequence itself may be short. Second, delay and complexity constraints may require processing of short source blocks. And finally, user experience may require low distortion when averaging over short blocks, which we term fidelity blocks. Existing work on the subject has largely coupled these issues, i.e., a source block is compressed taking into account the statistics of the distortion averaged over that block (usually the excess-distortion probability). For short source sequences, this coupling indeed makes sense. In this work, however, we instead consider the case of a long source sequence, whose compression performance is set by the interplay between the comparatively shorter processing and fidelity blocklengths. We focus on asymptotic analysis of the excess rate needed to ensure a given excess-distortion probability, for processing blocks that are shorter than the fidelity ones. Our main result is that the second-order performance (dispersion) is relatively unaffected by choosing a processing blocklength considerably shorter than the fidelity blocklength. Thus, one may use lower dimensional quantizers than existing work would otherwise suggest without sacrificing significant performance.

## I. INTRODUCTION

Finite-blocklength analysis of various source and channel information-theoretic settings has been a subject of great interest in the last years. But what exactly do we mean by “finite blocklength”? Various blocklength constraints, representing very different design considerations, can be imposed.

A rather generic finite-blocklength lossy compression scenario can be described as follows. A source emits a sequence of random symbols of length  $\ell$ , which we call the *source blocklength*. This sequence is fed into a source encoder. However, this encoder may be subject to different constraints, representing system delay requirements or encoding complexity. We consider a blockwise encoder which may only process jointly  $k$  source sam-

ples at a time.<sup>1</sup> We call  $k$  the *processing blocklength*. For each such processing block, the encoder chooses one of possible  $M$  descriptions of the source. At the destination, the decoder produces a lossy reconstruction of the source sequence, to be used by an observer. As this observer is oblivious to the internal workings of the encoding/decoding scheme, we measure its experience by a fidelity measure applied to a block of some length  $n$ , starting at position  $n_0$  (with respect to the start of the source sequence). We call  $n$  the *fidelity blocklength*. Thus, the performance of a scheme is characterized by the statistics of a fidelity function  $d(M, \ell, k, n, n_0)$ . By taking a worst-case or average starting point  $n_0$ , the fidelity reduces to a function of three blocklengths.

A natural goal is, to characterize the best performance achievable given a message set size  $M$  and the blocklengths  $\ell$ ,  $n$  and  $k$ . As a measure of performance, one may use single-letter characteristics of the distributions, such as the excess-distortion probability with respect to some prescribed threshold. We now note that this problem in fact covers scenarios of very different nature.

For example, one may be interested in short source sequences, that is,  $\ell < k, n$ . The source produces a sequence short enough for the encoder and the decoder to treat all symbols jointly, and also for the fidelity to be measured over the whole source and reconstruction sequences. In this case, there is only one relevant blocklength - the source blocklength. That is, one can take without loss of generality  $k = n = \ell$ . Indeed, previous work gives good understanding of the excess-distortion probability in this regime [1]–[3]

In this work, we concentrate on the opposite case, namely long source sequences:  $\ell \gg k, n$ . The source

<sup>1</sup>One may also consider a non-blockwise scheme limited by delay, or consider complexity directly. Such distinctions are beyond the scope of this work.

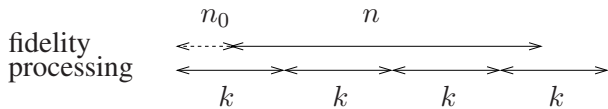


Fig. 1: Schematic view of the processing and fidelity blocks

produces a very long sequence, which is parsed by the encoder into many processing blocks, and also contains many fidelity blocks. We note that it is a very important scenario in practice. Consider the compression of a long video, for example. Practical encoders will not process the whole video jointly, but use a much shorter processing blocks. Also, the fidelity is not measured over the whole source block: lost frames in one part of the video cannot be compensated for by excellent-quality reproduction in another part.

In order to facilitate tractable analysis, we ignore edge effects of the source sequence, and take in this regime the source blocklength  $\ell$  to be infinite. Thus, we have two blocklengths at play: the encoder parses the infinite source sequence into blocks of length  $n$  for processing, while the fidelity is measured over blocks of length  $k$ .

Within the long sequence scenario, we choose to concentrate on the short-processing low-delay case  $k < n$ . After making some definitions, in Section III we derive the asymptotic behavior when both blocklength grow together. Then in Section IV, we consider growing fidelity blocklength as the processing blocklength remains constant, e.g. scalar quantization. Finally in Section V we discuss connections with channel and joint source-channel coding.

## II. DEFINITIONS

The source is an infinite i.i.d. sequence  $\dots, X_0, X_1, \dots$  where the symbols belong to some alphabet  $\mathcal{X}$  and have some distribution  $P$ . The encoder is a function  $\mathcal{X}^k \rightarrow \{1, \dots, M\}$ , applied to processing blocks  $X_{ak+1}, \dots, X_{(a+1)k}$  for integer  $a$ . The decoder is a function  $\{1, \dots, M\} \rightarrow \hat{\mathcal{X}}^k$ , used to reconstruct an infinite sequence  $\dots, \hat{X}_0, \hat{X}_1, \dots$  by placing each reconstructed processing block in the location of the original block. The fidelity is measured using a single-letter function  $d: \mathcal{X} \times \hat{\mathcal{X}} \rightarrow \mathbb{R}^+$ , averaged over blocks:

$$d_{n,n_0}(X, \hat{X}) = \frac{1}{n} \sum_{i=n_0+1}^{n_0+n} d(X_i, \hat{X}_i). \quad (1)$$

The parsing into processing and fidelity blocks is demonstrated in Figure 1.

The excess-distortion probability of a scheme with processing blocklength  $k$  and observation blocklength  $n$  is averaged over the offset between blocks:

$$p_e(n, k, D) = \frac{1}{k} \sum_{n_0=0}^{k-1} \Pr\{d_{n,n_0}(X, \hat{X}) > D\}. \quad (2)$$

The rate-distortion function (RDF) of a source with an i.i.d. distribution  $Q$  is given by  $R(Q, D)$ . If  $Q$  is the source distribution  $P$ , we simply write  $R(D)$ . The inverse function is denoted by  $D(R)$  or  $D(Q, R)$ .<sup>2</sup>

## III. GROWING PROCESSING BLOCKLENGTH

In this section, we let the processing blocklength  $k$  and fidelity blocklength  $n$  grow together, with  $k = o(n)$ .<sup>3</sup> We derive achievable rates as a function of  $k$  and  $n$  for a given excess-distortion probability.

We note two important results that are outside the region of interest, yet they can serve as benchmarks. For  $k = n$  we have the dispersion approximation. Let  $R(n, \epsilon, D)$  be the minimum rate needed to guarantee distortion at most  $D$  at blocklength  $n$  with excess-distortion probability  $\epsilon$ , then [2], [3]

$$R(n, \epsilon, D) = R(D) + \sqrt{\frac{V(D)}{n}} Q^{-1}(\epsilon) + O\left(\frac{\log n}{n}\right). \quad (3)$$

On the right hand side, the rate-distortion function is followed by a dispersion term, governed by the source dispersion  $V(D)$  and then a correction term. If we consider, rather than excess-distortion probability, the minimum rate of a  $k$ -dimensional code such that the *average* distortion is at most  $D$ , we have [4]:

$$R(k, D) = R(D) + \frac{\log k}{2k} + o\left(\frac{\log k}{k}\right). \quad (4)$$

Intuitively speaking, this can be seen as a limiting behavior of the excess-distortion rate when we take finite  $k$  but infinite  $n$ . Our main result below serves as a bridge between these two.

*Theorem 1:* Take a discrete memoryless source with distribution  $P$  and distortion level  $D$  s.t.  $V(D) > 0$ . Further assume that  $R(Q, D)$  is twice differentiable w.r.t.  $Q$  and  $D$  in some neighborhood of  $(P, D)$ . Let  $0 < \epsilon < 1$ . For any  $k = o(n)$  there exists a sequence of

<sup>2</sup>If the inverse is not unique at rate  $R$ , we take the lowest distortion satisfying the equality.

<sup>3</sup>Recall that we consider an infinite source sequence. Equivalently, we've first taken the source blocklength  $\ell$  to infinity, and only then we take the  $k$ - and  $n$ -limit.

schemes with processing and fidelity blocklengths  $k$  and  $n$ , respectively, and rates

$$R(n, k, \epsilon, D) = R(D) + \sqrt{\frac{V(D)}{n}} Q^{-1}(\epsilon) + O\left(\frac{\log k}{k}\right) \quad (5)$$

with excess-distortion probability at most  $\epsilon$ .

We note that the sequences  $k(n) = o(n)$  can be divided into the following two regimes.

1) Fidelity-limited regime. If

$$\frac{\log k}{k} = o\left(\sqrt{\frac{1}{n}}\right) \quad (6)$$

then the dispersion term remains as in the case  $k = n$  (3). That is, taking a shorter processing blocklength does not effect performance, to the dispersion approximation. Clearly, this dispersion is optimal.

2) Processing-limited regime. If (6) does not hold, then

$$R(n, k, \epsilon, D) = R(D) + O\left(\frac{\log k}{k}\right).$$

That is, to the first order of approximation the fidelity blocklength  $n$  takes no part, and the behavior is as if it was “infinite” (4). Thus, this behavior is optimal; however, we do not make a claim about the coefficient of the  $\log k/k$  term.

From the system-designer point of view, the existence of a non-trivial fidelity-limited regime is especially important. It means, that the same performance guaranteed by [2], [3] can be obtained (up to the dispersion approximation) even if the processing blocklength  $k$  is taken to grow almost as slow as  $\sqrt{n}$ .

The key in proving Theorem 1, is using a coding scheme that approaches the optimal distortion for the empirical statistics of the source within the processing block. The following gives the performance of such a scheme.

*Lemma 1:* For any discrete alphabets and distortion measure, denote by  $D(Q)$  the maximum distortion (averaged over the processing blocklength) given that the source type (over the same block) was  $Q$ . There exists a single scheme of processing blocklength  $k$  and rate  $R$ , such that:

$$D(Q) \leq D(R, Q) + O\left(\frac{\log k}{k}\right). \quad (7)$$

*proof outline:* The codebook consists of the union of per-type codebooks for all types over  $\mathcal{X}$ . By type-counting arguments, the rate of each codebook can be

$$R' = R - O\left(\frac{\log k}{k}\right). \quad (8)$$

By the refined type-covering lemma [5], there exists codebooks with maximum distortion  $D(Q)$  as long as

$$R' \geq R(Q, D(Q)) + O\left(\frac{\log k}{k}\right). \quad (9)$$

Combining the two we have:

$$R(Q, D(Q)) \geq R - O\left(\frac{\log k}{k}\right),$$

which is equivalent to the required result.

Now, the distortion over the fidelity block is an average over these processing-block distortions. the proof of Theorem 1 shows that we can use a linearization such that this is equivalent to averaging the type; then, performance is limited by the statistics of that type, measured over the whole fidelity blocks. The correction terms for this linearization all fall below the dispersion term, as long as  $k$  grows slow enough. We give the following proof outline.

Let  $\rho = n/k$ , and for simplicity assume that  $\rho$  is an integer. We have, then, a sequence of types  $Q_1, \dots, Q_\rho$ , and a sequence of distortions  $D_1, \dots, D_\rho$  according to (7).

By the smoothness assumptions of the theorems, we have for any  $Q \in \Omega_k(P)$ :

$$D(R, Q) = D(R, P) + \frac{(Q - P) \cdot \nabla R}{R'} + O\left(\frac{\log k}{k}\right) \quad (10)$$

where the redundancy term is uniformly bounded for types  $Q$  inside a neighborhood:  $\Omega_k(P) : \|P - Q\|^2 \leq \log k/k$ . Here,  $Q - P$  is a vector (of the alphabet size) of the difference between distributions  $Q$  and  $P$ , and  $\nabla R$  and  $R'$  are derivatives of the RDF w.r.t. the source distribution and distortion, respectively, evaluated at  $(P, D)$ .

Combining with (7) and applying to the processing blocks, we have:

$$D_i \leq D(R, P) + \frac{(Q_i - P) \cdot \nabla R}{R'} + O\left(\frac{\log k}{k}\right)$$

Now, let

$$\bar{Q} = \frac{1}{\rho} \sum_{i=1}^{\rho} Q_\rho$$

be the type over the observation block. Averaging, we can achieve distortion

$$D(\bar{Q}) \leq D(R, P) + \frac{(\bar{Q} - P) \cdot \nabla R}{R'} + O\left(\frac{\log k}{k}\right). \quad (11)$$

We need to ensure that at a rate  $R(n, k)$  the excess-distortion probability is at most  $\epsilon$ . By the union bound,

$$\epsilon \geq \rho \Pr\{Q_1 \notin \Omega_k(P)\} + \Pr\{\bar{Q} : D(\bar{Q}) > D\},$$

where the latter probability is bounded by (11), assuming that the type in all processing blocks was inside  $\Omega_k$ . Invoking a technical lemma from [2] stating that  $\Pr\{Q_1 \notin \Omega_k(P)\} = O(1/k^2)$ , and taking  $k$  as in the Theorem conditions, it remains to ensure that

$$\epsilon'(n) \triangleq \Pr\{\bar{Q} : D(\bar{Q}) > D\} \leq \epsilon - o\left(\sqrt{\frac{1}{n}}\right).$$

To that end, we can invert (11) back to terms of rate, and then note that we have a linearization that is in fact the first step in proving the rate-redundancy lemma in [2], with an additional  $O(\log k/k)$  redundancy term. Using that lemma we have the required rate with an  $O(\log n/n)$  term, which is dominated by the  $k$ -dependent one.

*Remark 1:* In the proof we have used the method of types, thus the correction terms grow with the alphabet size. This may not be necessary. As a matter of fact, the non-typewise analysis used to derive the dispersion in [3] uses a simple minimum-distortion encoder and a random codebook; thus, it is already a “best effort” quantizer as needed. However, in order to follow the proof technique we have used, one needs a better codebook ensemble, such that maximum distortion can be guaranteed deterministically for some class of source realizations, as in the “type-covering lemma”.

*Remark 2:* An important continuous-alphabet example that can be treated similar to the discrete case, is the quadratic-Gaussian one. An extension of Theorem 1 can be derived, using a quantizer that depends upon the empirical variance of the source block. Roughly speaking, the distortion in each  $k$ -block will be proportional to the empirical variance in that block, thus when averaging over the  $n$ -block we will also have distortion proportional to the empirical variance over that block. However, this derivation is somewhat more cumbersome, as the source has to be carefully divided into spherical shells (that is, the empirical variance quantized) in a judicious manner, as in done in e.g. [6].

*Remark 3:* Finally, we note that one may be interested in different asymptotics. Instead of fixing the excess-distortion probability, one may ask how that probability decays with the blocklengths for a fixed rate. For  $k = n$ , this leads to the excess-distortion exponent [1]. Adjusting Theorem 1 to this setting, one finds that also the exponent w.r.t.  $n$  remains unchanged even if  $k$  grows much slower.

## IV. SCALAR QUANTIZATION

In practice, many times scalar quantizers are used (that is, the processing blocklength is  $k = 1$ ), since the gain of vector quantization does not justify the complexity. We can still consider the fidelity-blocklength asymptotics of such schemes.

If we fix some quantizer size  $M$  and excess-distortion probability  $\epsilon$ , we can consider the behavior of the optimal distortion threshold  $D^*(M, n, \epsilon)$ .<sup>4</sup> Specifically, as in the previous section, we can ask about the dispersion. To that end, let  $\bar{D}(M)$  be the minimal expected distortion achievable by any scalar quantizer, and let  $V_{D, \min}(M)$  and  $V_{D, \max}(M)$  be the minimum and maximum (reps.) distortion variance of  $M$ -quantizers achieving  $\bar{D}(M)$ . Then we have the following.

*Proposition 1:*

$$D^*(M, n, \epsilon) = \bar{D}(M) + \sqrt{\frac{V_D(M)}{n}} Q^{-1}(\epsilon) + O\left(\frac{1}{n}\right),$$

where  $V_D(M) = V_{D, \min}(M)$  for  $\epsilon < 1/2$ ,  $V_D(M) = V_{D, \max}(M)$  otherwise.

*Proof outline:* These are the excess-distortion asymptotics of an average-distortion optimal scalar quantizer, by standard central-limit theorem (CLT) analysis of the i.i.d. sequence  $d(X_i, \hat{X}_i)$ . For any other quantizer the first term will be worse than  $\bar{D}(M)$ , thus for large enough  $n$  the performance cannot be better.

*Remark 4:* The dispersion terms above have the units of squared distortion, as opposed to squared rate units normally used in dispersion analysis. We work directly with distortion, since the quantizer size  $M$  is typically a small integer, unless high-rate quantization is concerned, thus considering small variations seems unnatural. In order to emphasize this difference, we have use the subscript  $D$ .

*Example 1:* One-bit scalar quantization in the quadratic-Gaussian case. Let the source be Gaussian with zero mean and unit variance, and let the distortion be measured by the squared reconstruction error. We take a scalar quantizer with  $M = 2$ . By symmetry, it suffices to consider reconstruction levels centered around zero. Denote these levels as  $\pm\gamma$ . Straightforward optimization (see e.g. [7]) that the expected distortion is minimized uniquely by the choice  $\gamma = \sqrt{2/\pi} \triangleq \gamma_\infty$ , yielding

$$\bar{D}(2) = 1 - \frac{2}{\pi}.$$

<sup>4</sup>The same analysis carries over to any other fixed finite  $k$ , by considering scalar quantization of super-symbols in  $\mathcal{X}^k$ . Indeed, in this case we need to average over the offset  $n_0$ , but this is insignificant for the sake of asymptotics.

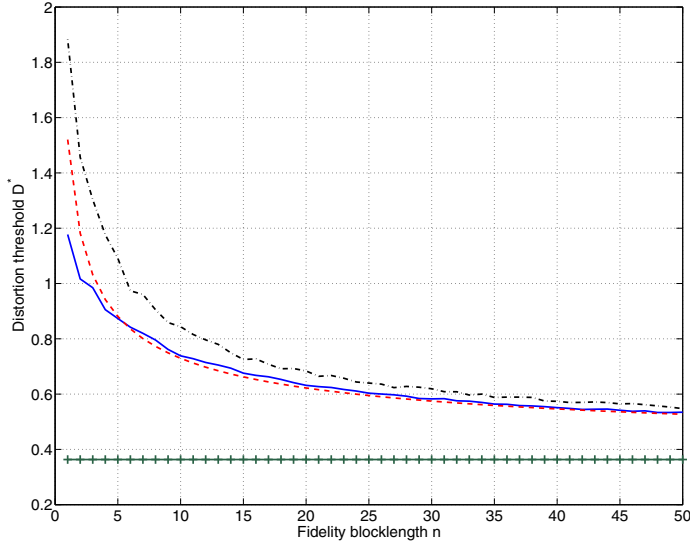


Fig. 2: One-bit scalar quantization in the quadratic-Gaussian case: distortion as a function of blocklength, with excess-distortion probability  $\epsilon = 0.03$ . The solid blue curve is the optimal distortion, while the dash-dotted black curve is the distortion achieved by using the mean-distortion optimal quantizer  $\gamma = \gamma_\infty$ . For reference, the dashed red curve is the dispersion approximation and the dotted green straight line shows the infinite-blocklength expected distortion.

The resulting dispersion is:

$$V_D(2) = 2 \int_{x=0}^{\infty} \left( (x - \gamma_\infty)^2 - \bar{D}(2) \right)^2 f_X(x) dx.$$

Beyond asymptotics, we can consider the exact performance at finite blocklength  $n$ , that is, fix some excess-distortion probability  $\epsilon$  and evaluate  $D^*(2, n, \epsilon)$ . We note, that the choice of quantizer (that is, of  $\gamma$ ) may vary with  $n$ . For  $n = 1$ , one should choose  $\gamma = \gamma_1$  where  $2Q(2\gamma_1) = \epsilon$ . For higher  $n$  analytic derivation becomes harder, and we resort to numerical optimization. The results, shown in Figure 2, demonstrate that indeed for short fidelity blocklength  $n$ , quantizer optimization can improve the excess-distortion performance.

## V. DISCUSSION: BEYOND SOURCE CODING

It is tempting to seek a channel-coding dual for the results in this work. However, usually in a channel setting a message error probability is considered, thus the error event is intimately related to the message blocklength and it is not clear how to treat the case where the message blocklength is much larger than the processing blocklength. In this context, it is worth to mention the constraint-length exponent [8], and in general the

distinction between blocklength and delay [9]. Further, an element that does introduce a non-trivial interplay between blocklengths is an input constraint. Similar to the discussion in the introduction, such a constraint comes from considerations that do not have to do with coding (e.g., the characteristics of power amplifiers, or limiting interference to other communication systems), thus they should be measured over some *resource block* of length  $m$ , in a manner oblivious to the parsing of information into processing blocks.

The joint source-channel coding setting is closer in spirit to this work; indeed, it subsumes channel coding with a bit-error rate criterion. We note, that for “matching” source-channel pairs [10], high-performance short-processing schemes may be more readily available. Without an input constraint, scalar schemes may have the same dispersion and exponent w.r.t. the fidelity blocklength  $n$ , as optimal schemes with  $k = n$  [11], [12]. Furthermore, for the quadratic-Gaussian case [12] proposes a scheme which is much in the spirit of Remark 2. We expect to extend these results, and show that for a large class of source-channel pairs, joint source-channel schemes may have a large advantage over separate ones when short processing blocklength is concerned.

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