Rateless Codes for the Gaussian Multiple Access Channel

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***Abstract*— We consider communication over the Gaussian multiple access channel (MAC) with unknown set of active users. The proposed multiple access strategy is distributed and achieves a maximum sum rate point on the boundary of the capacity region for this channel for any set of active users $\mathcal{S}$ simultaneously, as if $\mathcal{S}$ were known at the transmitters. The proposed coding scheme splits each user into a set of virtual users, each of which can be decoded using a single-user decoder at the receiver instead of having to decode all users jointly. We also present a generalization of this scheme to the case where the channel gains differ between users and each user only knows its own channel gain.**

I. Introduction and Motivation

Consider a set of users, each having a queue of messages that it wants to transmit through a wireless channel to some central base station. This base station is allowed to occasionally broadcast a short beacon signal to keep the users synchronized. Whenever a user detects the beacon signal, it starts transmitting a codeword corresponding to the first message in its queue, if it is not empty. As soon as the base station is able to decode all the messages currently being transmitted, it emits again a beacon, indicating that the current round of communication is over and starting the next one.

Note that this setup is rather decentralized and there are various degrees of uncertainty the users face. First, while each user knows if it currently has a message to transmit, it knows neither which nor how many other users are transmitting at the same time. Moreover, the channel gains can differ across users. Assuming the gain of the uplink and downlink channel are the same, as would be the case with, e.g., time division duplex, each user can infer the gain of its link from the strength of the beacon signal it received. The users are, however, ignorant of the channel gains of the other users. As the wireless medium is shared among all the users, these issues result in an uncertainty about the amount of interference each user will experience during any communication round.

We propose a multiple access strategy for this setup using successive decoding of the different users. That is, the decoder at the base station can use several single-user decoders in succession and does not need to decode all transmitting users jointly. At the same time, this scheme is optimal in the sense that it achieves the maximum possible sum rate for the case where the set of transmitting users as well as all the link qualities were known to all users. That is, neither the lack of knowledge about the link qualities or number and identity of interfering other users nor the requirement of using successive decoding at the base station result in a loss of achievable rate.

Throughout most of this paper, we assume that the single-user codes used in the successive decoding scheme above are (rateless) capacity achieving codes. This simplifies the analysis and allows to isolate the impact of successive decoding on the performance of the system. We will, however, briefly discuss how practical rateless single-user codes can be used within this architecture.

Rate-splitting is a technique for the multiple access channel (MAC) which is designed for use in conjunction with successive decoding [1]. In [1] both the number as well as the identity of the users is fixed and known to each user. The case where the identity of the users is not known, but their number is fixed and known, has been investigated in [2] and more recently in [3]. In there it is assumed that the channel gain is identical for each user. As the number of transmitting users is known, this implies that every user knows the amount of interference it will experience. This differs from the setup considered here, in which the unknown number as well as the unknown channel gains of interfering users result in uncertainty about the amount of interference. Queueing and scheduling aspects in a setup similar to the one described in this paper are analyzed in [4] and [5] respectively.

The remainder of this paper is organized as follows. In Section II, the problem setup is formalized, and some notation is introduced. Section III describes the proposed multiple access scheme and establishes the main result of this paper. Section IV contains concluding remarks.

II. Problem Formulation

The scenario of interest can be modelled as communication over a Gaussian MAC. We first consider the case where all the channel gains are equal to one. As will be shown in Section III-A, the general case can be reduced to this setup. There are K total users out of which a subset $\mathcal{S} \neq \emptyset$ is active. That is, only the users in the set $\mathcal{S}$ are using the channel. At time t (assumed to be discrete here) the received signal is

$$Y_t = \sum_{s \in \mathcal{S}} X_{s,t} + Z_t,$$

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where $\{Z_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. Gaussian random variables with zero mean and unit variance, independent of the channel inputs. The set $\mathcal{S}$ is known to the receiver; however, the transmitters know only whether they are in $\mathcal{S}$ or not (i.e., transmitter s only knows if $s \in \mathcal{S}$), but do not know the set $\mathcal{S}$ in general.

A code for this channel defines for every possible user s an encoding function (or codebook) $f_s : \{1, \ldots, M\} \to \mathbb{R}^{\infty}$ mapping the message m_s of user s into an infinite sequence of channel symbols $f_s(m_s)$ satisfying an average power constraint P. The index m_s of the codeword to be sent by transmitter s is chosen with uniform probability over the set $\{1, \ldots, M\}$. The receiver consists of $2^K - 1$ decoders, each of them for a specific set of active users $\mathcal{S}$.

Each of these decoders is specified by a (deterministic) decoding time $T_{\mathcal{S}}$ and a decoding function $\phi_{\mathcal{S}} : \mathbb{R}^{T_{\mathcal{S}}} \to \{1, \ldots, M\}^{|\mathcal{S}|}$ mapping the sequence $\{Y_t\}_{t=1}^{T_{\mathcal{S}}}$ into an estimate

$$\{\hat{m}_s\}_{s\in\mathcal{S}} \triangleq \phi_{\mathcal{S}}\Big(\{Y_t\}_{t=1}^{T_{\mathcal{S}}}\Big)$$

of the messages sent by each active user. The triple $\big(\{f_s\}_{s=1}^{K}, \{T_{\mathcal{S}}\}_{\mathcal{S}}, \{\phi_{\mathcal{S}}\}_{\mathcal{S}}\big)$ specifies a coding scheme.

The average probability of error $\bar{e}$ of a coding scheme is defined as

$$\bar{e} \triangleq \max_{\mathcal{S}} \bar{e}(\mathcal{S}),$$

$$\bar{e}(\mathcal{S}) \triangleq Pr\bigg[\bigcup_{s\in\mathcal{S}}\{\hat{m}_s \neq m_s\}\bigg|\mathcal{S}\bigg],$$

and the rate of communication if $\mathcal{S}$ is the set of active users is

$$R_{\mathcal{S}} \triangleq \frac{\log M}{T_{\mathcal{S}}}.$$

Every coding scheme results in a $2^K - 1$ tuple of rates $\{R_{\mathcal{S}}\}_{\mathcal{S}}$. A rate tuple $\{R_{\mathcal{S}}\}_{\mathcal{S}}$ will be called achievable if there exists a sequence of coding schemes indexed by M with rate converging to $\{R_{\mathcal{S}}\}_{\mathcal{S}}$ and such that $\lim_{M\to\infty} \bar{e} = 0$.

From a purely information theoretic point of view, the communication problem described above can be solved using standard results. Note first that the optimal input distribution for each active user s is the same for all sets $\mathcal{S}$ (namely, zero mean Gaussian with variance P). As, moreover, the number of different sets $\mathcal{S}$ is finite, there exists a code and a joint decoder achieving any rate

$$R_{\mathcal{S}} < C_{\mathcal{S}} \triangleq \frac{1}{2|\mathcal{S}|}\log\big(1 + |\mathcal{S}|P\big) \tag{1}$$

simultaneously for all $\mathcal{S}$ as if $\mathcal{S}$ were known at the transmitters. In other words, there exists a coding scheme operating for every $\mathcal{S}$ at the equal rate point on the boundary of the capacity region of the ordinary Gaussian MAC with known $\mathcal{S}$ at the transmitters as defined for example in [6].

The coding scheme described in the last paragraph has, however, the disadvantage of requiring a joint decoder. That is, in the worst case K codes have to be decoded jointly, resulting in an unacceptably large decoding complexity. In the next section, we will describe a lower complexity coding scheme, building on the rate-splitting approach [1] and its generalization in [2], which allows the use of multiple single-user decoders at the receiver instead of a joint decoder.

III. Rateless Codes

In this section, we construct a rateless code which uses several single-user decoders at the receiver instead of one joint decoder. We require this coding scheme to achieve rates arbitrarily close to $C_{\mathcal{S}}$ simultaneously for all sets $\mathcal{S}$ of active users. In other words, we want the decoding times $T_{\mathcal{S}}$ to satisfy

$$T_{\mathcal{S}} \leq \frac{\log M}{C_{\mathcal{S}} - \varepsilon} \tag{2}$$

for an arbitrary $\varepsilon > 0$ and M large enough. Observe from (1) and (2) that the requirement on $T_{\mathcal{S}}$ depends on $\mathcal{S}$ only through $|\mathcal{S}|$.

We split each user s into L virtual users. For each s, virtual user l uses a codebook with M_l codewords. We require that $\prod_{l=1}^{L} M_l = M$, i.e., that the total number of messages for each user s is M. The codewords of virtual user l of user s are created as infinite length sequences of independent Gaussian random variables with mean zero and variance $P_l(t)$ for the t-th symbol in the sequence, subject to the constraint that

$$\sum_{l=1}^{L} P_l(t) = P \tag{3}$$

for all times t. A power allocation $\{P_l(t)\}_{l,t}$ will be called valid if $P_l(t) \geq 0$ for all t, l and (3) is satisfied for all t. The codewords of all virtual users of each user s are added to form the codeword of that user. At time $T_{\mathcal{S}}$, the decoder $\phi_{\mathcal{S}}$ first decodes layer L of virtual users for all $s \in \mathcal{S}$, regarding all other virtual users (of the same and all other users in $\mathcal{S}$) as noise. The decoded codewords of the virtual users in layer L are then subtracted from the received signal, and the decoder continues in the same manner with layer $L-1$ of virtual users until all virtual users have been decoded. Hence the decoder $\phi_{\mathcal{S}}$ consists of $L|\mathcal{S}|$ single-user decoders, which are used successively.

Note that for finite L this decoding procedure is suboptimal, as it regards all codewords of virtual users in the same layer as noise, whereas some of them could have been subtracted off the received signal after decoding. It does allow, however, to choose the encoders $\{f_s\}_{s=1}^{K}$ to be identical for each user s, which is necessary due to the distributed nature of the problem. Moreover, we show in the sequel that this scheme can approach optimality as the number of virtual users L goes to infinity.

We first consider power allocations that are *constant across time* (i.e., $P_l(t) = P_l$ for all virtual users l and times t). Lemma 1 shows that as $M \to \infty$ and $L \to \infty$ a power allocation $\{P_l\}_l$ can be found such that for every possible set of users $\mathcal{S}$ there exists a decoding time $T_{\mathcal{S}}$ and a splitting of messages $\{M_l\}_l$ with the following two properties: All messages can be reliably decoded at time $T_{\mathcal{S}}$, and the rate $R_{\mathcal{S}}$ is close to $C_{\mathcal{S}}$ in the sense of (2). Note that the choice of $\{M_l\}_l$ is allowed to depend on the set of active users $\mathcal{S}$. As in an actual system $\{M_l\}_l$ would have to be chosen before communication begins and without knowing $\mathcal{S}$, the power allocation given by Lemma 1 can not be used to guarantee optimal communication for all possible sets $\mathcal{S}$ simultaneously. This problem is addressed in Theorem 2. It states that as $M \to \infty$ and $L \to \infty$ a *time varying* power allocation $\{P_l(t)\}_{l,t}$ and a splitting of messages $\{M_l\}_l$ can be found such that for every possible set of users $\mathcal{S}$ there exists a decoding time $T_{\mathcal{S}}$ with the following two properties: All messages can be reliably decoded at time $T_{\mathcal{S}}$, and the rate $R_{\mathcal{S}}$ is close to $C_{\mathcal{S}}$ in the sense of (2).

The first lemma generalizes and strengthens a result from [2]; part of the proof follows along the lines of the one there.

Lemma 1. *For a valid $\{P_l\}_l$ and $\mathcal{S}$ there exists $T_{\mathcal{S}}$ and $\{M_l\}_l$ such that $\lim_{M\to\infty} \bar{e}(\mathcal{S}) = 0$ and*

$$\lim_{L\to\infty} \lim_{M\to\infty} R_{\mathcal{S}} = C_{\mathcal{S}},$$

if and only if $\lim_{L\to\infty} P_l = 0$ for all $l \in \mathbb{N}$. Moreover, for any finite L

$$C_{\mathcal{S}} - \lim_{M\to\infty} R_{\mathcal{S}} \le \frac{P}{4} \sup_l P_l \left(|S| + \frac{1}{1-P_l} \right).$$

Proof. Call $R_{\mathcal{S}}$ the rate of the above scheme as $M \to \infty$ and $L \to \infty$. Using genie aided decoder arguments as in [1], the highest achievable rate with successive decoding as described above is

$$\lim_{L\to\infty} \lim_{M\to\infty} R_{\mathcal{S}} = \lim_{L\to\infty} \sum_{l=1}^{L} \frac{1}{2} \log(1 + x_l),$$

where

$$x_l \triangleq \frac{P_l}{1 + (|\mathcal{S}| - 1)P_l + |\mathcal{S}| \sum_{i=1}^{l-1} P_i}.$$

Consider

$$\begin{aligned}\sum_{l=1}^{L} \frac{1}{2} \log(1 + x_l) &= \frac{1}{2} \sum_{l=1}^{L} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x_l^k \\ &= \frac{1}{2} \sum_{l=1}^{L} x_l + \frac{1}{2} \sum_{l=1}^{L} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} x_l^k. \quad (4)\end{aligned}$$

For the second term, we get

$$\begin{aligned}\left| \frac{1}{2} \sum_{l=1}^{L} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} x_l^k \right| &\le \frac{1}{2} \sum_{l=1}^{L} \sum_{k=2}^{\infty} \frac{1}{k} P_l^k \\ &\le \frac{1}{4} \sum_{l=1}^{L} P_l^2 \sum_{k=0}^{\infty} P_l^k \\ &= \frac{1}{4} \sum_{l=1}^{L} P_l \frac{P_l}{1-P_l} \\ &\le \frac{P}{4} \sup_l \frac{P_l}{1-P_l}, \quad (5)\end{aligned}$$

which converges to zero if $\lim_{L\to\infty} P_l = 0$ for all $l \in \mathbb{N}$. For the first term in (4) we have

$$x_l \ge \frac{P_l}{1 + |\mathcal{S}| \sum_{i=1}^{l} P_i}$$

and hence if $\lim_{L\to\infty} P_l = 0$ for all $l \in \mathbb{N}$

$$\begin{aligned}\lim_{L\to\infty} \lim_{M\to\infty} R_{\mathcal{S}} &= \lim_{L\to\infty} \frac{1}{2} \sum_{l=1}^{L} x_l \\ &\ge \lim_{L\to\infty} \frac{1}{2} \sum_{l=1}^{L} \frac{P_l}{1 + |\mathcal{S}| \sum_{i=1}^{l} P_i} \\ &= \frac{1}{2} \int_{y=0}^{P} \frac{1}{1 + |\mathcal{S}| y} dy \quad (6) \\ &= \frac{1}{2|\mathcal{S}|} \log(1 + |\mathcal{S}| P) \\ &= C_{\mathcal{S}}.\end{aligned}$$

For finite L, the approximation error in (6) is upper bounded by

$$\begin{aligned}&\sum_{l=1}^{L} \frac{1}{4} P_l^2 \max_{y\in[0,P]} \left| \frac{d}{dy} \frac{1}{1 + |S| y} \right| \\ &= \sum_{l=1}^{L} \frac{1}{4} P_l^2 \max_{y\in[0,P]} \frac{|S|}{\left(1 + |S| y\right)^2} \\ &= \sum_{l=1}^{L} \frac{|S|}{4} P_l^2 \\ &\le \frac{P|S|}{4} \sup_l P_l.\end{aligned}$$

Together with (5) this implies that

$$\begin{aligned}C_{\mathcal{S}} - \lim_{M\to\infty} R_{\mathcal{S}} &\le \frac{P|S|}{4} \sup_l P_l + \frac{P}{4} \sup_l \frac{P_l}{1-P_l} \\ &\le \frac{P}{4} \sup_l P_l \left(|S| + \frac{1}{1-P_l} \right).\end{aligned}$$

Conversely, if there exists some l such that $P_l \ge \delta > 0$ then it is easily seen that just by decoding this layer suboptimally, we will always get a rate $R_{\mathcal{S}}$ strictly below $C_{\mathcal{S}}$ for all L. □

As the codewords for each virtual user are a sequence of independent random variables and as the channel is memoryless, the direct part of Lemma 1 can be seen to apply also to the case where the power allocation is time varying. We use this kind of power allocation in the following. More precisely, we choose powers $P_l(t)$ constant on each time interval $\{1, T_1\}, \{T_1 + 1, T_2\}, \ldots, \{T_{K-1} + 1, T_K\}$ where $T_k \triangleq T_{\mathcal{S}}$ for any $\mathcal{S}$ such that $|\mathcal{S}| = k$. We denote the power for virtual user l in time interval k by $P_l(T_k)$. The splitting of the messages is done uniformly, i.e., we set $M_l = M^{1/L}$ for all l.

For the first time interval, choose $P_l(T_1)$ such that

$$\frac{1}{2} \log \left(1 + \frac{P_l(T_1)}{1 + \sum_{i=1}^{l-1} P_i(T_1)} \right) = \frac{R_1}{L}$$

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for all l and some constant R_1. For subsequent time intervals, we allocate powers $P_l(T_k)$ such that

$$\sum_{j=1}^{k} \frac{\Delta T_j}{2T_k} \log\left(1+\frac{P_l(T_j)}{1+(k-1)P_l(T_j)+k\sum_{i=1}^{l-1}P_i(T_j)}\right) = \frac{R_k}{L} \quad (7)$$

for all l, constants R_k and with $\Delta T_j \triangleq T_j - T_{j-1}$, $T_0 \triangleq 0$. From Lemma 1, it is clear that if a valid power allocation of this type exists such that all powers $P_l(T_j)$ converge to zero as $L \to \infty$, then we can, for any k, make R_k arbitrarily close to $C_\mathcal{S}$ for all $\mathcal{S}$ such that $|\mathcal{S}| = k$ by choosing M and L large enough. The next theorem establishes that a power allocation of this type exists and hence that the above scheme is asymptotically in M and L optimal for all sets of active users $\mathcal{S}$ simultaneously. Even though the theorem only claims existence of such a power allocation, its proof is constructive in the sense that it gives an algorithm to find such an allocation.

Theorem 2. *For $M_l = M^{1/L}$ there exists a valid $\{P_l(T_k)\}_{l,k}$ and $\{T_\mathcal{S}\}_\mathcal{S}$ such that $\lim_{M\to\infty} \bar{e} = 0$ and for all $\mathcal{S}$*

$$\lim_{L\to\infty}\lim_{M\to\infty} R_\mathcal{S} = C_\mathcal{S}.$$

Proof. Identify R_k in (7) as $R_\mathcal{S}$ for any $\mathcal{S}$ such that $|\mathcal{S}| = k$. Using Lemma 1, we then only have to show that a valid power allocation satisfying (7) exists and that $\lim_{L\to\infty} P_l(T_k) = 0$ for all l and k.

Note first that $\lim_{L\to\infty} R_k/L = 0$ for any power allocation satisfying (7). That is, the rate in each layer of virtual users goes to zero. Assume then that there exists a $\delta > 0$ such that for all L we have $P_l(T_k) \geq \delta$ for at least one $l \in \{1,\dots,L\}$ and some $k \in \{1,\dots,K\}$. Then we get from (7)

$$\frac{R_k}{L} \geq \frac{\Delta T_k}{2T_k}\log\left(1+\frac{P_l(T_k)}{1+(k-1)P_l(T_k)+k\sum_{i=1}^{l-1}P_i(T_k)}\right)$$

which is bounded away from zero. Hence there exists some sequence $k(L)$ such that $\lim_{L\to\infty} R_{k(L)}/L > 0$, contradicting the fact that all layers have the same rate. Thus if a valid power allocation satisfying (7) exists it must satisfy $\lim_{L\to\infty} P_l(T_k) = 0$ for all $l \in \mathbb{N}$ and $k \in \{1,\dots,K\}$.

We will show by induction that such a valid power allocation exists. Clearly, we can find a power allocation such that (7) is satisfied for $k = 1$ with $T_1 = \log(M)/(C_{\{s\}} - \varepsilon)$ for some $\varepsilon > 0$. Assume then we have fixed decoding times $\{T_j\}_{j=1}^{k-1}$ and that the result holds up to that point. That is, we have chosen valid powers $P_l(T_j)$ for $j \in \{1,\dots,k-1\}$ such that for each layer l (7) is satisfied for some R_{k-1}. Define

$$\delta_l(k) \triangleq \frac{T_{k-1}R_{k-1}}{L} - \sum_{j=1}^{k-1}\frac{\Delta T_j}{2}\log\left(1+\frac{P_l(T_j)}{1+(k-1)P_l(T_j)+k\sum_{i=1}^{l-1}P_i(T_j)}\right).$$

By the induction hypothesis $\delta_l(k) \geq 0$ for all $l \in \{1,\dots,L\}$. Using the fact that $T_{k-1}R_{k-1} = T_k R_k$, we get from (7) that we have to find a power allocation such that

$$\frac{\Delta T_k}{2}\log\left(1+\frac{P_l(T_k)}{1+(k-1)P_l(T_k)+k\sum_{i=1}^{l-1}P_i(T_k)}\right) = \delta_l(k).$$

Solving for $P_l(T_k)$, we find

$$P_l(T_k) = \left[\frac{\exp\big(2\delta_l(k)/\Delta T_k\big)-1}{1-\Big(\exp\big(2\delta_l(k)/\Delta T_k\big)-1\Big)(k-1)}\right] \times \left[1+k\sum_{i=1}^{l-1}P_i(T_k)\right]. \quad (8)$$

Note that $\exp\big(2\delta_l(k)/\Delta T_k\big) - 1 \geq 0$. Consider now $\sum_{l=1}^{L} P_l(T_k)$ as a function of ΔT_k as defined through (8). Let B be the greatest value of ΔT_k such that

$$1-\Big(\exp\big(2\delta_l(k)/\Delta T_k\big)-1\Big)(k-1) \geq 0,$$

holds for all $l \in \{1,\dots,L\}$ with equality for at least one l. It is easily checked that $B > 0$. $\sum_{l=1}^{L} P_l(T_k)$ is a continuous function of ΔT_k over (B,∞) with $\lim_{\Delta T_k\to B}\sum_{l=1}^{L} P_l(T_k) = \infty$ and $\lim_{\Delta T_k\to\infty}\sum_{l=1}^{L} P_l(T_k) = 0$. Hence there exists ΔT_k^* such that $\sum_{l=1}^{L} P_l(T_k) = P$. Moreover, for $\Delta T_k \in (B,\infty)$, we have $P_l(T_k) \geq 0$ for all $l \in \{1,\dots,L\}$. Hence choosing $\Delta T_k = \Delta T_k^*$ results in a valid power allocation for time slot k. Even though ΔT_k^* is in general not an integer, we can make the rounding error as small as desired by choosing M large enough. This concludes the induction step. □

Figure 1 shows the fraction of capacity $C_\mathcal{S}$ achievable with this scheme as a function of number of layers L used. As expected the number of layers needed to achieve a fixed fraction of capacity increases with the number of active users $|\mathcal{S}|$. It can, however, be observed from the same figure that, even for $|\mathcal{S}|$ as large as 20, only a moderate number of layers is needed to operate at a rate close to capacity. For example, for $|\mathcal{S}| = 20$ and 20 layers, more than 90% of the sum capacity can be achieved. Note also that the rate loss incurred for a fixed finite number of layers L depends only on the number of active users $|\mathcal{S}|$, and not on the number of potential active users K. In other words, the system can be designed very conservatively (assuming a large number of potential users K) without having to pay a penalty (at least in terms of achievable rate).

A. Arbitrary Channel Gains

In this section, we remove the restriction that all channel gains are equal to one. That is, each user has now an associated (constant) channel gain. As each user knows its own channel gain, this is equivalent to imposing a possibly different average power constraint $\tilde{P}_s$ for each user $s \in \mathcal{S}$. We assume that for all $s \in \mathcal{S}$ we have $\tilde{P}_s = \tilde{L}_s P$ for some $\tilde{L}_s \in \mathbb{N}$. Split each user s into $\tilde{L}_s$ virtual users. This results in a set $\tilde{\mathcal{S}}$ of active virtual users each with identical power constraint P. We can now use the coding scheme described in the last section to

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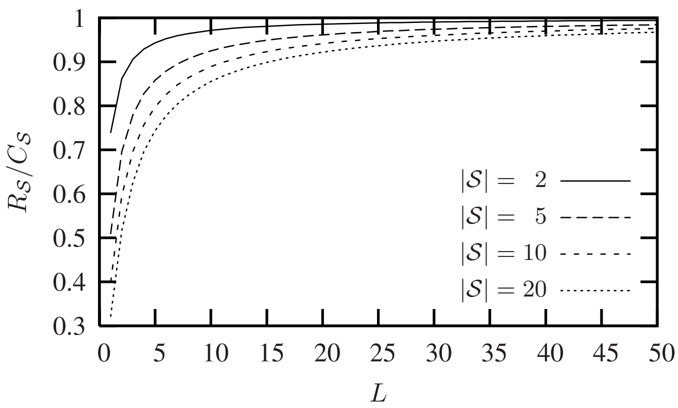

Fig. 1. Fraction of capacity $C_{\mathcal{S}}$ achievable as a function of number of layers (virtual users) L for different numbers of active users $|\mathcal{S}|$. The nominal SNR (i.e., without interference) is 0 dB.

achieve a rate $R_{\tilde{\mathcal{S}}}$ arbitrarily close to $C_{\tilde{\mathcal{S}}}$. Hence each user $s \in \mathcal{S}$ can transmit at a rate arbitrarily close to

$$\tilde{L}_s C_{\tilde{\mathcal{S}}} = \frac{\tilde{L}_s}{2|\tilde{\mathcal{S}}|} \log\big(1 + |\tilde{\mathcal{S}}|P\big).$$

We have

$$\sum_{s\in\mathcal{S}} \tilde{L}_s C_{\tilde{\mathcal{S}}} = \frac{1}{2}\log\big(1 + |\tilde{\mathcal{S}}|P\big) = \frac{1}{2}\log\left(1 + \sum_{s\in\mathcal{S}} \tilde{P}_s\right),$$

and hence the above scheme achieves a maximum sum rate point on the boundary of the achievable rate region of the Gaussian MAC.

Note that with this procedure not all users are communicating at the same rate. As each user is split into a number of virtual users proportional to its average power constraint, this implies that the relative rate a user can communicate at is also proportional to its average power constraint. For example, when two users are active both with equal average power constraint P they are both able to communicate at the same rate, say R. When two users are active, but this time the first user has an average power constraint of P and the second user has a constraint of $2P$, the first user is able to communicate at some rate $\tilde{R}$ whereas the second user achieves a rate $2\tilde{R}$. Note that $\tilde{R} \neq R$ in general in this situation.

There is also a tradeoff between the accuracy P with which different power levels can be approximated and the number of virtual users $\tilde{L}_s$ we need to split each user into. If P is chosen very small, $\tilde{L}_s$ will be very large for many users s, resulting in a higher decoding complexity. As, however, in any practical system the actual channel gains (or, equivalently, the power constraints in our setup) can only be estimated within a certain precision, a nonzero P can always be chosen such that the approximation error is negligible compared to the estimation error.

B. Practical Rateless Codes

Up to this point, we have assumed that every virtual user uses a capacity achieving single-user code. More precisely, the codewords of virtual user l of user s are created as infinite length sequences of independent Gaussian random variables with mean zero and variance $P_l(t)$ for the t-th symbol in the sequence. In a real system, we have to replace this with a practical rateless code satisfying a time varying power constraint.

The design of such a code is simplified by the fact that the power allocated to each virtual user vanishes as the number of these users grows. This makes it possible to use simple practical codes and efficient decoding techniques described in [7]. These codes are constructed from a single good "base" code of a fixed blocklength (say n) for the standard additive white Gaussian noise channel with constant power constraint. The codeword corresponding to message m in the rateless code is constructed by repeating a scaled and "dithered" version of the codeword corresponding to message m in the base code. Decoding is performed by combining blocks of length n of the received sequence into a single vector of length n, which is then decoded using the decoder for the base code. For a detailed description of this construction and an analysis of its performance, see [7].

IV. Conclusion

We have described a communication scheme for the Gaussian MAC, achieving a maximum sum rate point on the boundary of the capacity region for this channel for any set of active users $\mathcal{S}$ simultaneously, even when $\mathcal{S}$ is unknown at the transmitters. The proposed coding scheme splits each user into a set of virtual users, each of which can be decoded using a single-user decoder at the receiver instead of having to decode all users jointly. The presented solution also generalizes easily to the case where the channel gains differ between users and each user only knows its own channel gain.

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