Achieving the Full MIMO Diversity-Multiplexing Frontier with Rotation-Based Space-Time Codes*

Huan Yao
Lincoln Laboratory
Massachusetts Institute of Technology
Lexington, MA 02420
yaohuan@ll.mit.edu

Gregory W. Wornell
Dept. EECS
Massachusetts Institute of Technology
Cambridge, MA 02139
gww@mit.edu

Abstract

The recently established diversity-multiplexing frontier characterizes the high-SNR tradeoff between the best possible robustness and throughput gains obtainable by employing multiple antennas for digital communication in fading environments.

We focus on the case of two transmit and at least two receive antennas, and show that a sufficient condition for a family of codebooks indexed by rate to achieve the full diversity-multiplexing frontier is that the worst-case codeword difference determinant either does not decay to zero with rate or decays subexponentially. We further provide a constructive proof that the full frontier is achievable by codes of length two by developing families of rotation-based space-time block codes with rate-independent nonzero worst-case determinants. These determinants are optimized over reasonably rich code subspaces. By contrast, Gaussian codes of this length were known not to achieve the frontier, and earlier rotation-based codes were known only to achieve the extremal points of the frontier.

Simulation results also verify that the new codes have sufficient coding gain to achieve error rates close enough to the infinite length code outage bound, and thus they provide superior performance to Alamouti's orthogonal space-time block code at spectral efficiencies beyond about 4 b/s/Hz.

1 Introduction and Background

It is well-known that using multiple antennas to create a multiple-input multiple-output (MIMO) channel can significantly increase the robustness and throughput of communication systems in fading environments. For a given number of antennas, there is a continuous tradeoff between these gains, which Zheng and Tse [15] quantify in terms of a high SNR characterization of all diversity-multiplexing gain pairs achievable with sufficiently long codes.[1] The boundary of this set of pairs is the diversity-multiplexing

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[1]These notions of diversity and multiplexing gains should not be confused with other definitions in the literature. The notion of interest in this paper are summarized in Section 2.

frontier. And while infinitely long codes are generally needed to achieve the best coding gains, only finite length codes are required to achieve the frontier. In particular, for the case of two transmit and at least two receive antennas, which is the focus of this paper, Gaussian codes can achieve the frontier if they have at least length-three, but cannot if the length is two [15].[2]

Among structured block codes, it is known [15] that the popular orthogonal space-time block codes (OSTBC) in general [11], and Alamouti's two transmit antenna code in particular [1], only achieve a single point on the frontier: they achieve maximum diversity but sacrifice multiplexing gain in order to achieve very low decoding complexity.

As a result, there have been a variety of other structured code designs proposed to achieve simultaneously good diversity and multiplexing performance. For example, [6, 9] build numerically optimized codebooks from linear combinations of basis matrices, though at higher rates this optimization becomes impractical.

Another class of designs is based on applying unitary transformations to multidimensional QAM constellations, a construction that dates back at least to Lang [7]. For convenience, we refer to such rotation-based constructions as "tilted-QAM" codes. Such codes were used in [13, 2, 5] for single-antenna communication over multiple fades and later in [3, 8] for multiple-antenna communication within a single fade — the case of interest in this paper. The tilted-QAM constructions of these papers achieve the two end-points of the diversity-multiplexing frontier, and are optimized using algebraic number theoretic tools.

In this paper we show that by careful choice of the unitary transformation one can obtain tilted-QAM codes of length two that achieve the full diversity-multiplexing frontier. In particular, we develop a sufficient condition for any family of codes to achieve the full frontier, then develop and optimize tilted-QAM codes that meet this condition.

Our system model is as follows. The multiple antenna channel of interest has two transmit antennas, at least two receive antennas, and experiences flat Rayleigh fading. In our model, the channel remain constant over blocks longer than the code length T so that each codeword experiences only one channel realization. Moreover, the channel is known at the receiver but not at the transmitter.

When the transmission rate is R b/s/Hz, there are 2^{RT} codeword matrices to be designed. More specifically, we use the system model $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}$, where $\mathbf{H}$ is the 2×2 channel matrix with independent, identically-distributed entries, each with a zero-mean, unit-variance, circularly-symmetric complex Gaussian distribution $\mathcal{CN}(0, 1)$, $\mathbf{X}$ is the $2 \times T$ transmitted signal matrix, $\mathbf{W}$ is independent, identically-distributed noise with distribution $\mathcal{CN}(0, 2\sigma_w^2)$, and $\mathbf{Y}$ is the received signal matrix.

2 The Diversity-Multiplexing Frontier

We begin by summarizing those aspects of the frontier relevant to this paper, together with the associated notation. For a given SNR and SNR-dependent transmission rate $R(\text{SNR})$, we let $P_e(R(\text{SNR}), \text{SNR})$ denote the system (block) error probability. Diversity and multiplexing gains are defined as how fast error probability decays and rate increases, respectively, with SNR in the high SNR limit, i.e.,

$$d = -\lim_{\text{SNR}\to\infty} \frac{\log P_e(\text{SNR})}{\log \text{SNR}} \quad \text{and} \quad r = \lim_{\text{SNR}\to\infty} \frac{R(\text{SNR})}{\log_2 \text{SNR}}. \tag{1}$$

[2]The rank criterion in [11] precludes length-one codes from reaching the frontier.

Since the analysis focuses on the high-SNR regime, the following asymptotic relation notation [15] is convenient: the relation $f \doteq g$ for two functions f and g denotes exponential equality, i.e., $\lim_{\mathrm{SNR}\to\infty} \log f(\mathrm{SNR})/\log \mathrm{SNR} = \lim_{\mathrm{SNR}\to\infty} \log g(\mathrm{SNR})/\log \mathrm{SNR}$. For example, using this notation, (1) can be expressed as $P_e(\mathrm{SNR}) \doteq \mathrm{SNR}^{-d}$ and $2^R \doteq \mathrm{SNR}^r$. The relations $\dot{\geq}$ and $\dot{\leq}$ are defined analogously. This notation allows us to focus on exponential growth rates alone and neglect other aspects of expressions.

The diversity-multiplexing frontier for two transmit and two receive antennas is the piecewise linear characteristic depicted in Fig. 1(a) [15]. Note that the maximum achievable diversity gain is four, and the maximum achievable multiplexing gain is two. To obtain this frontier, it suffices to note that when arbitrarily long codes are used, only outage events contribute to the error probability, so $P_e = P_{\mathrm{out}}(R(\mathrm{SNR}), \mathrm{SNR}) = \Pr[C(\mathbf{H}, \mathrm{SNR}) < R(\mathrm{SNR})]$, where

$$C(\mathbf{H}, \mathrm{SNR}) = \log_2 \left(|\det(\mathbf{H})|^2 \left(\frac{\mathrm{SNR}}{2} \right)^2 + \|\mathbf{H}\|^2 \frac{\mathrm{SNR}}{2} + 1 \right) \tag{2}$$

is the channel mutual information with an independent, identically-distributed Gaussian input distribution. In (2), the norm $\|\cdot\|$ is defined via $\|A\|^2 = \sum_{i,j} |a_{ij}|^2$.

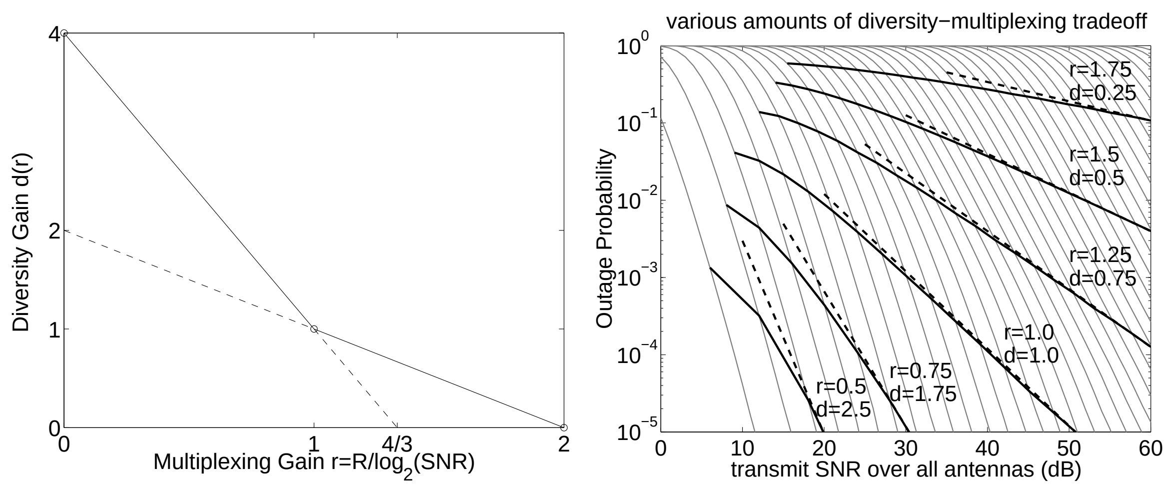

Figure 1: (a) Diversity-multiplexing frontier for the two-transmit, two-receive antenna case. (b) Interpreting the frontier operating points by plotting $P_{\mathrm{out}}(R(\mathrm{SNR}), \mathrm{SNR})$ for fixed $R = 1, 2, \ldots$ with light curves and $R = r \log_2(\mathrm{SNR})$ with dark curves.

The operating points on the frontier can be interpreted via Fig. 1(b), where the limiting error probability (i.e., outage) is plotted as a function of SNR for different rates R using light curves. To obtain the end-points of the frontier, we note that at sufficiently low outage probability (i.e., high SNR), the limiting slope of each such curve is four, which is the maximum diversity gain achievable for fixed rates, i.e., $r = 0$. And for a fixed P_{out}, at sufficiently high SNR, the horizontal separation approaches $r = 2$ b/s/Hz per 3 dB, which is the maximum multiplexing gain. To obtain the intermediate points on the frontier, we evaluate $P_{\mathrm{out}}(R = r \log_2(\mathrm{SNR}), \mathrm{SNR})$ as a function of SNR for fixed r, obtaining the dark curves. From these curves we see that the smaller values of r corresponding to slower rate increases with SNR yield sharper asymptotic outage probability decays. The limiting slope of each curve, indicated by a dashed line, represents the diversity gain of the associated frontier operating point. We emphasize each dark curve represents the performance of a family of codes indexed by rate, since rate is increasing with SNR along these curves.

3 A Sufficient Condition for Frontier Achievability

We now establish a sufficient condition for achieving the full diversity-multiplexing frontier.

Theorem 1 *For a system with two transmit and at least two receive antennas and a code length $T \geq 2$, consider a family of codebooks indexed by rate R that is scaled such that the peak and average codeword energy grow with R as $E_s \doteq \max_{\mathbf{X}} \|\mathbf{X}\|^2 \doteq 2^{R/2}$. Then a sufficient condition for achieving the diversity-multiplexing frontier is*

$$\min_{\mathbf{X_1} \neq \mathbf{X_2}} |\det(\mathbf{X_1} - \mathbf{X_2})| \dot{\geq} 1. \tag{3}$$

Before proceeding with a sketch of the proof, we make two remarks. First, (3) is equivalent to the condition that either the worst-case codeword difference determinants do not decay to zero with rate or decay at most subexponentially. Second, while the sufficient condition of the theorem neglects constants since they don't affect achieving the frontier, such constants do affect the ultimate coding gain in general, and larger determinants are preferred. Indeed, since determinants appear in pairwise error probabilities as coefficients of the SNR^{-d} term, maximizing determinant is related to maximizing coding gain [11].

Proof: It suffices to examine the error probability behavior in the high SNR regime. Our approach is to first upper bound the error probability conditioned on a particular realization of $\mathbf{H}$, then average over all realizations. To bound the conditional error probability, we lower bound the asymptotic minimum distance when the channel is not in outage and show that this distance is large relative to the noise.

To develop our bound, we begin by letting $\mathbf{\Delta} \stackrel{\text{def}}{=} \mathbf{X}_1 - \mathbf{X}_2$ denote the difference between two arbitrary codewords, and lower bound $\|\mathbf{H\Delta}\|$ in terms of $\det(\mathbf{H})$ and $\|\mathbf{H}\|$, which both appear in the (2). Specifically, letting $\lambda_{\max} \geq \lambda_{\min}$ denote the singular values of $\mathbf{\Delta}$, and using both the energy normalization and (3), we have

$$\left.\begin{array}{c} \lambda_{\max}^2 + \lambda_{\min}^2 = \|\mathbf{\Delta}\|^2 \dot{\leq} \max_{\mathbf{X}} \|\mathbf{X}\|^2 \doteq \mathrm{SNR}^{r/2} \\ \lambda_{\max}^2 \lambda_{\min}^2 = |\det(\mathbf{\Delta})|^2 \dot{\geq} 1 \doteq \mathrm{SNR}^0 \end{array}\right\} \Longrightarrow \left\{ \begin{array}{ccccc} \mathrm{SNR}^0 & \dot{\leq} & \lambda_{\max}^2 & \dot{\leq} & \mathrm{SNR}^{r/2} \\ \mathrm{SNR}^{-r/2} & \dot{\leq} & \lambda_{\min}^2 & \dot{\leq} & \mathrm{SNR}^{r/2} \end{array}\right. . \tag{4}$$

To lower bound $\|\mathbf{H\Delta}\|^2$ using $|\det(\mathbf{H})|$, we use (3) [3]

$$\|\mathbf{H\Delta}\|^2 \geq 2|\det(\mathbf{H\Delta})| = 2 \cdot |\det(\mathbf{H})| \cdot |\det(\mathbf{\Delta})| \dot{\geq} |\det(\mathbf{H})|. \tag{5}$$

To lower bound $\|\mathbf{H\Delta}\|^2$ using $\|\mathbf{H}\|^2$, it suffices to note that $\mathbf{H}$ must be scaled by at least $\lambda_{\min}$, whence

$$\|\mathbf{H\Delta}\|^2 \geq \lambda_{\min}^2 \|\mathbf{H}\|^2 \dot{\geq} \mathrm{SNR}^{-r/2} \|\mathbf{H}\|^2. \tag{6}$$

[3] When there are more than 2 receive antennas or $T > 2$, and matrices are generally not square, $\det(\cdot)$ is defined as the product of the two largest singular values.

Combining (5) and (6) we obtain the normalized minimum distance

$$\frac{\|\mathbf{H}\boldsymbol{\Delta}\|^2}{\sigma_w^2} \dot{\geq} \max\Big(\text{SNR}^{1-r/2}|\det(\mathbf{H})|, \text{SNR}^{1-r}\|\mathbf{H}\|^2\Big), \tag{7}$$

where using the energy normalization in the form $E_s \doteq 2^{R/2} \doteq \text{SNR}^{r/2}$ we have expressed the noise in the form $\sigma_w^2 \doteq E_s/\text{SNR} \doteq \text{SNR}^{r/2-1}$.

In turn, since the channel mutual information (2) can be written in the form

$$2^{C(\mathbf{H})-R} \doteq \Big(\text{SNR}^{1-r/2}|\det(\mathbf{H})|\Big)^2 + \text{SNR}^{1-r}\|\mathbf{H}\|^2. \tag{8}$$

we see that when the channel is not in outage $|\det(\mathbf{H})|$ and $\|\mathbf{H}\|^2$ can not both be small and, hence, $\|\mathbf{H}\boldsymbol{\Delta}\|^2$ cannot be small relative to σ_w^2. More precisely, we have

$$C(\mathbf{H}) > R \Longrightarrow \left\{ \begin{array}{c} \text{SNR}^{1-r/2}|\det(\mathbf{H})| \dot{\geq} 1 \\ \text{or } \text{SNR}^{1-r}\|\mathbf{H}\|^2 \dot{\geq} 1 \end{array} \right\} \Longrightarrow \frac{\|\mathbf{H}\boldsymbol{\Delta}\|^2}{\sigma_w^2} \dot{\geq} 1. \tag{9}$$

From the lower bound (9) on the distances between the received codewords when there is no outage, we can analytically upper bound the associated conditional error probability $\Pr[\text{error}|\mathbf{H}]$ using the Erlang distribution of $\|\mathbf{W}\|^2$. Further integration over the Rayleigh distribution of $\mathbf{H}$ leads to the conclusion that the error probability decays as the outage probability bound. In the interests of space, further details are deferred to [14]. ■

4 Frontier-Achieving Tilted-QAM Codes

We now develop a tilted-QAM code meeting the condition of Theorem 1. Since optimizing a tilted-QAM code over all possible unitary transformations involves a large number of parameters, we consider optimizations over simpler subclasses of such transformations.

We begin by considering transformations where the diagonal and back-diagonal entries of $\mathbf{X}$ are rotated independently. And while [3] and [8] consider particular subclasses of complex rotations for such purposes, we restrict our attention to the real subclass and show that they are sufficient. Later we also show that relatively small improvements in determinant are obtainable by considering yet a different subclass of complex rotations.

In our code design, four information symbols, $s_{ij} \in \mathbb{Z}+\mathbb{Z}j$, are encoded into one codeword matrix $\mathbf{X}$ as follows:

$$\begin{bmatrix} x_{11} \\ x_{22} \end{bmatrix} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{22} \end{bmatrix} \text{ and } \begin{bmatrix} x_{21} \\ x_{12} \end{bmatrix} = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} \begin{bmatrix} s_{21} \\ s_{12} \end{bmatrix}. \tag{10}$$

Note that when we choose the constellation size to match the target rate, i.e., $M^2 = 2^{R/2}$, the required energy normalization of Theorem 1 follows automatically from the fact that $s_{ij} \in \mathbb{Z}+\mathbb{Z}j$ when the codewords are drawn uniformly.

As we will now show, it is possible to choose the rotation angles for this class of codes such that the worst-case determinant does not decay to zero and, moreover, is *independent* of rate.[4] As such, it meets the condition (3) for achieving the full frontier, and has the added implementation advantage that there is a universally good pair of rotation angles.

[4] Codebooks with this determinant property have also been earlier developed for single-antenna communication over multiple fades [2, 5].

Theorem 2 *For codeword matrices defined in* (10), *and any QAM constellation carved from* $\mathbb{Z} + \mathbb{Z}j$, *the maximum worst-case determinant of difference matrices is*

$$\max_{(\theta_1,\theta_2)} \min_{\mathbf{X}_1 \neq \mathbf{X}_2} |\det(\mathbf{X}_1 - \mathbf{X}_2)| = \frac{1}{2\sqrt{5}}, \tag{11}$$

and achieved by $(\hat{\theta}_1, \hat{\theta}_2) = \left(\frac{1}{2}\arctan(\frac{1}{2}), \frac{1}{2}\arctan(2)\right)$.

Proof: Let δ_{ij} be the differences of the information symbols used to encode $\mathbf{X}_1$ and $\mathbf{X}_2$, so that entries of $\mathbf{\Delta} = \mathbf{X}_1 - \mathbf{X}_2$ are related to $\delta_{ij} \in \mathbb{Z} + \mathbb{Z}j$ according to (10), and

$$2\det(\mathbf{\Delta}) = \sin(2\theta_1)(\delta_{11}^2 - \delta_{22}^2) + 2\cos(2\theta_1)\delta_{11}\delta_{22} - \sin(2\theta_2)(\delta_{21}^2 - \delta_{12}^2) - 2\cos(2\theta_2)\delta_{12}\delta_{21}. \tag{12}$$

For binary constellation, by listing all $\det(\mathbf{\Delta})$ expressions for all δ_{ij} 4-tuples, we can easily show that $(\hat{\theta}_1, \hat{\theta}_2)$ and its symmetric variations are optimal, with which the worst-case determinant is $1/(2\sqrt{5})$ and is obtained at $(\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}) = (1, 0, 0, 0)$, among others. As constellation grows, the worst-case determinant can only decrease or remain constant, since there are additional codewords to minimize over. So to prove Theorem 2, it suffices to show that using $(\hat{\theta}_1, \hat{\theta}_2)$, $1/(2\sqrt{5})$ is actually achievable for arbitrarily large constellations , i.e., $|\det(\mathbf{\Delta})| \geq 1/(2\sqrt{5})$ for all nonzero 4-tuples of $\delta_{ij} \in \mathbb{Z} + \mathbb{Z}j$.

Substituting $(\hat{\theta}_1, \hat{\theta}_2)$ into (12), we have,

$$J \stackrel{\text{def}}{=} 2\sqrt{5}\det(\mathbf{\Delta}) = \delta_{11}^2 - \delta_{22}^2 + 4\delta_{11}\delta_{22} + 2\delta_{12}^2 - 2\delta_{21}^2 - 2\delta_{21}\delta_{12}. \tag{13}$$

Since $\delta_{ij} \in \mathbb{Z} + \mathbb{Z}j$, so is J. Now we need to prove the following.

Lemma 1 *For* $\delta_{ij} \in \mathbb{Z} + \mathbb{Z}j$, $J = 0$ *only when* $\delta_{11} = \delta_{12} = \delta_{21} = \delta_{22} = 0$.

Let us perform some completion of squares and change of variables. Let $a \stackrel{\text{def}}{=} \delta_{11} + 2\delta_{22}$, $b \stackrel{\text{def}}{=} \delta_{22}$, $c \stackrel{\text{def}}{=} 2\delta_{12} - \delta_{21}$, and $d \stackrel{\text{def}}{=} \delta_{21}$, then $2J = 4\sqrt{5}\det(\mathbf{\Delta}) = 2a^2 - 10b^2 + c^2 - 5d^2$.

To prove Lemma 1 and Theorem 2, it is sufficient to prove the following.

Lemma 2 *For* $a, b, c, d \in \mathbb{Z} + \mathbb{Z}j$, $2a^2 + c^2 = 5(2b^2 + d^2)$ *only if* $a = b = c = d = 0$.

This in turn requires the following lemma.

Lemma 3 *For* $x, y \in \mathbb{Z} + \mathbb{Z}j$, *if* $5|2x^2 + y^2$, *then* $5|x$, $5|y$, *and* $25|2x^2 + y^2$. [5]

Proof: Let $x = 5q_x + r_x$ and $y = 5q_y + r_y$, such that, $r_x, r_y \in \{0, 1, 2, 3, 4\} + \{0, 1, 2, 3, 4\}j$ and $q_x, q_y \in \mathbb{Z} + \mathbb{Z}j$. $5|2x^2 + y^2$ implies $5|2r_x^2 + r_y^2$. It is straight forward to verify that the only case where $5|2r_x^2 + r_y^2$ is $r_x = r_y = 0$. Therefore, $5|x, 5|y$, and $25|2x^2 + y^2$. ■

Returning to Lemma 2 and applying Lemma 3 to $2a^2 + c^2 = 5\,(2b^2 + d^2)$ yields

$$5|2a^2 + c^2 \Rightarrow 5|a,\ 5|c,\ 25|2a^2 + c^2 \Rightarrow 5|2b^2 + d^2 \Rightarrow 5|b,\ 5|d,\ 25|2b^2 + d^2. \tag{14}$$

Since a, b, c, and d are all multiples of 5, we can divide them all by 5 and obtain an essentially identical equation, $2a'^2 + c'^2 = 5\left(2b'^2 + d'^2\right)$, where $a', b', c', d' \in \mathbb{Z} + \mathbb{Z}j$. We can repeat the above argument and divide by 5 indefinitely. Thus, the only possible solution is $a = b = c = d = 0$, which implies $\delta_{11} = \delta_{12} = \delta_{21} = \delta_{22} = 0$. Lemma 2, Lemma 1 and Theorem 2 are all now complete. ■

[5] For complex integers, divisibility by a real integer (denoted by |) is defined as both real and imaginary parts being divisible.

Pairwise Error Probability Analysis

It is worth remarking that these tilted-QAM codes can also be analyzed in terms of pairwise error probabilities, which is useful when the overall error probability can be approximated by the product of the number of worst-case codeword pairs and their pairwise error probability. In the scenario of interest, the pairwise error probability $\Pr[\mathbf{X}_1 \to \mathbf{X}_2]$ can be upper bounded by [14]

$$\Pr[\mathbf{X}_1 \to \mathbf{X}_2] \dot{\leq} \left(\prod_{i=\min}^{\max} \left(1 + \frac{\lambda_i^2}{\sigma_w^2} \right) \right)^{-2} \dot{\leq} \left(\prod_{i=\min}^{\max} \left(\lambda_i^2 \mathrm{SNR}^{1-r/2} \right) \right)^{-2} \doteq \frac{\mathrm{SNR}^{2r-4}}{|\det(\boldsymbol{\Delta})|^4}, \tag{15}$$

from which we see that the worst codeword pairs are those with the worst-case determinant of $1/(2\sqrt{5})$. We conjecture that there are on the order of $M^4 \doteq \mathrm{SNR}^r$ many such codeword pairs [14], in which case the total error probability is $P_e \dot{\leq} \mathrm{SNR}^{2r-4}\mathrm{SNR}^r \doteq \mathrm{SNR}^{3r-4}$. Indeed, this is consistent with the diversity-multiplexing frontier for $0 \leq r \leq 1$; for $1 \leq r \leq 2$, the upper bound (15) is overly aggressive.

Sensitivity of Rotation Angles

One might expect that the sensitivity of our codes to the rotation angle should increase with constellation size, since points further from the origin are affected more by any rotation. This intuition is in fact correct, and has likely affected numerical searches for good tilted-QAM codes in the past [3]. The sensitivity is readily seen in Fig. 2, where we plot the worst-case determinant as a function of θ_1 with $\theta_2 = \pi/4 - \theta_1$ for 4-QAM, 9-QAM, and 16-QAM constellations. While the peak is obviously fixed, it gets sharper with increasing rate, so that small deviations can significantly impact worst-case determinants. Nevertheless, the numerical results imply that it should not be difficult in practice to obtain sufficient precision for typical constellations such as 4-QAM and 16-QAM provided some care is taken in the system implementation.

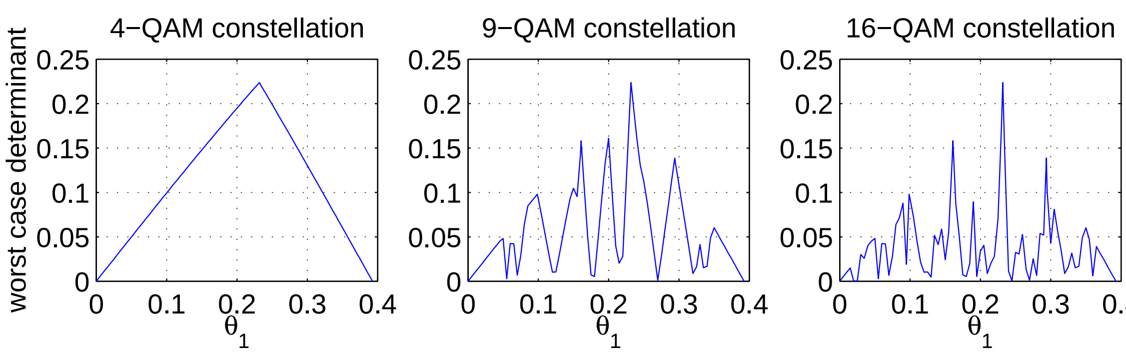

Figure 2: Worst-case determinant as a function of rotation angle θ_1 where $\theta_2 = \pi/4 - \theta_1$.

Fig. 2 also suggests that there are other rotation angles leading to smaller but still constant and nonzero worst-case determinants. For example, angles $\theta_1 = \arctan(1/3)/2$, $\arctan(2/3)/2$, and $\arctan(1/5)/2$, with $\theta_2 = \pi/4 - \theta_1$, yield constant worst-case determinants of $1/(2\sqrt{10})$, $1/(2\sqrt{13})$ and $1/(2\sqrt{26})$, respectively. Hence, these codes also all achieve the full diversity-multiplexing frontier, though can be expected to achieve less coding gain.

Codes From Complex Rotations

Modest improvements in determinant are possible with other parameterizations of the unitary transformation in the tilted-QAM code. For example, building on the codes developed in [5, 3, 8], we show codes of the subclass

$$\begin{bmatrix} x_{11} \\ x_{22} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{j\pi/4} \\ 1 & -e^{j\pi/4} \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_{21} \\ x_{12} \end{bmatrix} = \frac{e^{j\theta}}{\sqrt{2}} \begin{bmatrix} 1 & e^{j\pi/4} \\ 1 & -e^{j\pi/4} \end{bmatrix} \begin{bmatrix} s_{21} \\ s_{12} \end{bmatrix}, \quad (16)$$

offer such improvements. Indeed, with $\theta = \pi/12$, for 4-QAM constellation, the worst-case determinant improves to $\sin(\pi/12) = 0.2588 > 0.2236 = 1/(2\sqrt{5})$ is achieved, which corresponds to a 0.6 dB gain in SNR. Moreover, with $e^{j\theta} = (3+j)/\sqrt{10}$, the worst-case determinant of $1/(2\sqrt{5})$ is again achieved for all constellation sizes. The proof involves a change of variables and employing Lemma 2. Still larger determinants may arise from considering other subclasses of unitary transformations.

5 Simulation Results and Coding Gain

In this section, we numerically evaluate the coding gain associated with the new tilted-QAM codes, and show that the resulting gain is sufficient to approach the outage probability limit quite closely even at moderate SNR.

Since the encoder transformation and the fading matrix can be combined into a linear composite channel, the maximum likelihood decoding we use in this section can be efficiently implemented using sphere decoding techniques [12]. It should be emphasized that performance is in general sacrificed if the minimum distance decoder is not used. For example, if it were replaced with a lattice decoder, which does not take into account of the constellation boundaries, then the optimal diversity-multiplexing tradeoff is not be achieved [14].

Fig. 3(a) depicts the average block error rate as a function of SNR using dark curves, along with light curves showing outage probability behavior that correspond to the infinite length code performance bound. From this comparison, we see that the block error rate curves have slopes and spacings consistent with our analysis. In particular, for a fixed error probability, at sufficiently high rate (or SNR), the separation between the curves is 6 dB, which corresponds to the maximum multiplexing gain of $r = 2$ b/s/Hz per 3 dB. And, for a given rate, at sufficiently low error probability, the slope of each curve approaches four, which is the maximum diversity gain. Moreover, the gaps to the outage bounds are quite small, reflecting the fact that our tilted-QAM code performs quite well even at moderate SNR.

For comparison, Fig. 3(b) depicts the corresponding performance of Alamouti's OSTBC [1], which is an orthogonalized repetition code that encodes two information symbols into one 2×2 codeword matrix. While we see that the slope of each curve approaches four at high SNR, the multiplexing gain that is sacrificed for low complexity decoding results in the curves being separated by 12 dB or $r = 1$ b/s/Hz per 3 dB.

Comparing Figs. 3(a) and (b), we see that they are similar at 4 b/s/Hz. Below that, Alamouti's OSTBC is effectively optimal and is therefore preferred for its low decoding complexity. As rate increases, our tilted-QAM code increasingly outperforms OSTBC due to the superior multiplexing gain, allowing the new tilted-QAM code to achieve the same rates at much lower SNR.

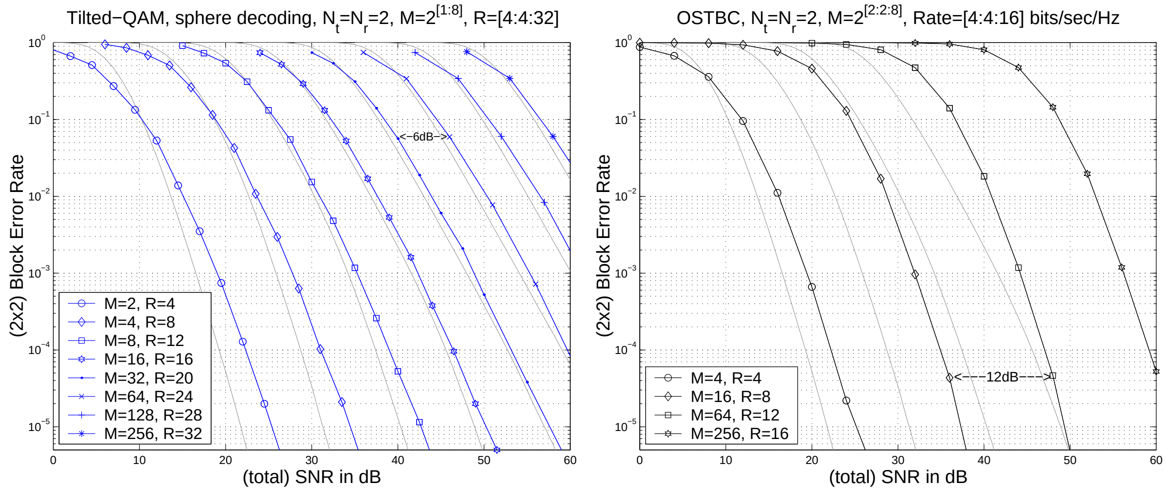

Figure 3: (a) Tilted-QAM code performance at rates $R = 2N_t \log_2(M) = 4, 8, 12, \ldots$ b/s/Hz using M^2-QAM constellations of sizes $M = 2, 4, 8, \ldots$. (b) OSTBC performance at rates $R = 2\log_2(M) = 4, 8, 12, 16$ b/s/Hz with $M = 4, 16, 64, 256$. Corresponding outage probability curves are superimposed for comparison.

As a final remark, the coding gain of our tilted-QAM codes can naturally be further enhanced through concatenation with other traditional error-correction codes [14]. Our simulations show that using hard-decision based Reed-Solomon code, the gap to outage probability is 5.2 dB at 10^{-2} (long) block error rate. Alternatively, using LDPC codes with a suitably designed iterative decoder exploiting soft-decision information, the gap is reduced to 3.1 dB for $R = 6$ b/s/Hz.

6 Concluding Remarks

We have shown that for codes of length two, structured codes can achieve the full diversity-multiplexing frontier, unlike Gaussian codes. Morever, these codes take the convenient form of a tilted-QAM structure with a suitably chosen rate-independent rotation matrix. While more general classes of rotation matrices can be considered, we would expect at most modest further improvements in coding gain.

Not surprisingly, code design becomes more involved when there are more than two antennas at encoder and decoder. This more general problem is studied in [8] and [4]. Both use an architecture where a codeword matrix is divided into diagonal and circularly-wrapped layers, resembling a cyclic version of the well-known BLAST architecture. An open question is how to design such a code to ensure subexponential worst-case determinant decay. The proposal in [8] promises nonzero determinants, but gives no guarantee on the magnitude. The scheme in [4] uses codes originally designed for the single-antenna multiple-fade problem, which is a reasonable starting point, but not sufficient, as the authors pointed out. However, solutions sufficient for practical purposes may not be so difficult to obtain. For example, when there are four transmit and receive antennas, one could factor the system into two separate two-transmit, four-receive antenna subsystems and employ joint decoding. Such a system would achieve a maximum diversity of 8 rather than 16, but is probably good enough for typical SNR regimes when the target error rate is in the neighborhood of 10^{-6}.

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