

# A CLASS OF STOCHASTIC RESONANCE SYSTEMS FOR SIGNAL PROCESSING APPLICATIONS

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## ABSTRACT

In this paper, the class of stochastic resonance systems based on  $M$ -level quantizer maps is developed. We derive expressions for the output invariant density and autocorrelation function of these maps when they are driven by a square wave in noise. These systems are shown to provide signal-to-noise ratio enhancement and robustness to noise characteristics. These properties render the quantizer maps potentially appealing for a wide range of signal processing applications such as interference suppression and robust communication. A framework for the analysis of more general discrete-time stochastic resonance systems is also presented, which is based on approximating these systems via quantizer maps.

## 1. INTRODUCTION

Stochastic resonance is a phenomenon encountered in certain bistable nonlinear systems (*i.e.*, systems with two stable points) when driven by a periodic signal in noise. Specifically, for certain signal strength levels, and in the absence of noise (or at small noise levels) the system output shows a small-amplitude oscillation, while remaining in the vicinity of a particular stable point. As the noise level is increased, however, a regular large-amplitude oscillation of the output between the two stable modes occurs at the period of the drive. Naturally, these systems often display an enhancement of output signal-to-noise ratio (SNR) in a given input SNR regime. For this reason, such systems are appealing candidates for use in a variety of engineering contexts. In terms of signal analysis, such systems constitute potentially useful models for natural phenomena such as the regularity of appearance of earth's ice ages [1], as well as for detection mechanisms in certain species, such as predator sensing by crayfish [2]. In terms of signal synthesis, the induced SNR enhancement renders them attractive in a number of applications in signal communication and processing, such as robust communication and interference suppression. In order to exploit the phenomenon of stochastic resonance in such applications, there is a need for tools to analyze these systems in the presence of various forms and degrees of distortion.

Several aspects of continuous-time systems exhibiting stochastic resonance have been explored in the literature; see *e.g.*, [3] [4]. A variety of heuristic algorithms have also

been proposed for using stochastic resonance systems as detection devices in backgrounds of additive, stationary white Gaussian noise given varying degrees of *a priori* information; see *e.g.*, [5] [6]. However, many of the characteristics of such systems are not well understood.

In this paper, we focus our attention on the class of first-order, discrete-time systems whose dynamics are governed by an  $M$ -level quantizer. For noisy periodic inputs, we derive analytic expressions describing the evolution of the output probability distribution as a function of time and the output autocorrelation function, which are important quantities for evaluating the degree of stochastic resonance exhibited by these systems. The system corresponding to  $M = 2$  is the discrete-time counterpart of a Schmitt trigger device, which has been shown to exhibit stochastic resonance behavior [6] [7]. For this map we also derive in closed-form the phase-averaged autocorrelation function. This function conveniently decomposes into signal and noise components, suggesting a natural definition for the output SNR. The analysis of the systems for  $M = 2$ , in turn, leads to a method for solving for the autocorrelation function of the output for  $M > 2$ , which we explore. The class of systems based on  $M$ -level quantizers capture the rich structure and important features of more general stochastic resonance systems, and they are potentially important in a range of signal synthesis applications. Furthermore, this class of systems can be used to approximate to arbitrary accuracy any of a much broader class of stochastic resonance systems, governed by continuous maps. As an example, we obtain the output invariant density for a map governed by a continuous function, which is the discrete-time counterpart of a well-known bistable continuous-time system exhibiting stochastic resonance.

## 2. $M$ -LEVEL QUANTIZER MAPS DRIVEN BY PERIODIC SIGNALS IN NOISE

The sequences  $y[n]$  of interest in this work are generated according to the following one-dimensional dynamics

$$y[n+1] = F(y[n] + x[n]), \quad (1)$$

where  $x[n]$  is a drive signal, and  $F(\cdot)$  is an  $M$ -level uniform quantizer. Typically,  $F(\cdot)$  is an approximation of a smooth function  $\tilde{F}(\cdot)$ . The quantizer map  $F(\cdot)$  can be conveniently described in the form

$$F(x) = \sum_{\ell=0}^{M-1} Y_{\ell} I_{\Delta_{\ell}}(x), \quad (2)$$

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where  $I_{\Delta}(\cdot)$  is the indicator function on  $\Delta$ ,  $\Delta_{\ell} = (X_{\ell}, X_{\ell+1}]$  for  $0 \leq \ell < M-1$ , and  $\Delta_{M-1} = (X_{M-1}, \infty)$ , and where the  $Y_{\ell}$ 's are the quantizer levels, satisfying  $Y_{\ell} < Y_{\ell+1}$ , for all  $\ell$ . We also restrict our attention to functions  $F(\cdot)$  that are odd, so that the analysis presented in this paper applies to approximations of any odd bistable or multi-stable mappings  $\tilde{F}(\cdot)$ . For the remainder of this paper, unless we specify otherwise, we will be considering  $\tilde{F}(\cdot)$  corresponding to the piecewise-linear soft-limiter map

$$\tilde{F}(x) = \begin{cases} \gamma x & |x| \leq L/\gamma \\ L \operatorname{sgn} x & \text{otherwise.} \end{cases}$$

In this case, the quantizer level spacing is given by  $\delta = 2/(M-1)$ , the quantizer levels are given by  $Y_{\ell} = L(\ell\delta - 1)$  for  $0 \leq \ell < M$ , and  $X_{\ell} = L[(\ell - 1/2)\delta - 1]/\gamma$  for  $1 \leq \ell < M$ ,  $X_0 = -\infty$ . Without loss of generality, we restrict our attention to the case  $L = 1$ .

We assume throughout that the mappings of interest are driven by a periodic signal in noise, *i.e.*,

$$x[n] = s[n] + w[n], \quad (3)$$

where  $s[n]$  is a period- $N$  signal and  $w[n]$  is an i.i.d. evenly-distributed noise sequence. The output  $y[n]$  of system (1) with  $F(\cdot)$  given by (2) and driven by  $x[n]$  in (3) has a first-order probability density that is time-varying in  $n$ . A particularly useful representation for  $y[n]$  generated by the system (1) driven by  $x[n]$  in (3), is based on the two-dimensional equivalent model

$$y[n+1] = F(y[n] + g(\theta[n]) + w[n]) \quad (4a)$$

$$\theta[n+1] = \theta[n] + \omega_o, \quad (4b)$$

where  $g(\cdot)$  is a  $2\pi$ -periodic function, and  $\omega_o = 2\pi/N$ . In this paper, we focus on the case where  $s[n] = g(\theta[n])$  is a discrete-time square wave of amplitude  $A$ , *i.e.*,

$$g(\theta) = \begin{cases} -A & -\pi \leq \theta < 0 \\ A & 0 \leq \theta < \pi, \end{cases} \quad (5)$$

and  $\theta[0]$  is uniformly distributed in  $(-\pi, \pi]$ . Via this formulation, the invariant density for  $y[n]$  corresponding to (4) can be used to determine the sequence of densities for mappings of the form (1) as a function of  $n$ .

### 3. INVARIANT DENSITY FOR QUANTIZER MAPS

In order to compute the output statistics of systems of the form (4), the invariant density of the pair  $(y, \theta)$  is required. This distribution can be subsequently used to obtain the periodic sequence of densities of  $y[n]$  corresponding to the one-dimensional map (1) driven by a square wave, where  $\theta[0]$  is known.

**Proposition 1** Consider the map (4) with  $F(\cdot)$  given by (2) and  $g(\cdot)$  given by (5), and let  $\mathbf{B}(\theta)$  denote the  $M \times M$  probability transition matrix for the output, *i.e.*,

$$[\mathbf{B}(\theta)]_{i,j} = p_{y[n+1]|y[n],\theta[n]}(y[n+1] = Y_i | y[n] = Y_j, \theta[n] = \theta).$$

Let  $p_i[n] = p(Y_i, \omega_o n)$ , where  $p(y, \theta)$  is the invariant density for the map (4). Then for  $M \geq 2$ :

$$p_i[n] = \begin{cases} \sum_j k_{i,j} \lambda_j^n & -\frac{N}{2} \leq n \leq 0 \\ \sum_j \tilde{k}_{i,j} \lambda_j^n & 0 \leq n \leq \frac{N}{2} \end{cases} \quad (6)$$

where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M$  are the eigenvalues of  $\mathbf{B}(0)$  ( $\mathbf{B}(\pi)$ ), the  $\mathbf{k}_j$ 's ( $\tilde{\mathbf{k}}_j$ 's) are the associated eigenvectors of  $\mathbf{B}(0)$  ( $\mathbf{B}(\pi)$ ) suitably scaled, and  $k_{i,j} = [\mathbf{k}_j]_i$  ( $\tilde{k}_{i,j} = [\tilde{\mathbf{k}}_j]_i$ ). In particular, for  $M = 2$ , we have  $\lambda_1 = 1$ ,  $\lambda_2 = 1 - \sum_i \alpha_i(0)$  and

$$\begin{aligned} k_{1,1} &= c\alpha_1(0), & k_{1,2} &= c(\alpha_1(0) - \alpha_1(\pi))/(1 + \lambda_2^{N/2}), \\ \tilde{k}_{1,1} &= c\alpha_1(\pi), & \tilde{k}_{1,2} &= -k_{1,2}\lambda_2^{N/2}, \end{aligned}$$

where  $c = (2\pi(1 - \lambda_2))^{-1}$ , and  $\alpha_i(\theta) = [\mathbf{B}(\theta)]_{1-i,i}$ , denotes the escape probability from state  $Y_i$  as a function of  $\theta$ .

*Proof:* Clearly,  $\lambda_1 = 1$  since  $\mathbf{B}(\theta)$  is a probability transition matrix. Next, using  $G_F\{\cdot\}$  to denote the density evolution operator for the system (4), we have

$$\begin{aligned} p_{y[n+1],\theta[n+1]}(y, \theta) &= G_F\{p_{y[n],\theta[n]}(y, \theta)\} \\ &= \sum_k P(y; k, \theta - \omega_o) p_{y[n],\theta[n]}(k, \theta - \omega_o) \end{aligned} \quad (7)$$

where  $P(y; k, \theta) = p_{y[n+1]|y[n],\theta[n]}(y|k, \theta)$  is the kernel associated with  $G_F\{\cdot\}$ , and

$$P(y; k, \theta) = \Pr[(w[n] + k + g(\theta)) \in F^{-1}(\{y\})].$$

Let  $\mathbf{q}(\theta) = [p(Y_0, \theta), p(Y_1, \theta), \dots, p(Y_{M-1}, \theta)]^T$ . By  $N$  recursive applications of (7), we obtain

$$\tilde{\mathbf{q}}(\theta) = \tilde{\mathbf{B}}(\theta) \tilde{\mathbf{q}}(\theta) \quad (8)$$

where  $\tilde{\mathbf{q}}(\theta) = [\mathbf{q}(\theta)^T, \mathbf{q}(\theta - \omega_o)^T, \dots, \mathbf{q}(\theta - (N-1)\omega_o)^T]^T$ , and where the  $(k, j)$ -th  $M \times M$  block of  $\tilde{\mathbf{B}}(\theta)$  is given by

$$[\tilde{\mathbf{B}}(\theta)]_{k,j} = \begin{cases} \mathbf{B}(\theta - k\omega_o) & \text{if } k = j + 1 \bmod N \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

It follows readily from (5) that

$$\mathbf{B}(\theta_1 - k\omega_o) = \mathbf{B}(\theta_2 - k\omega_o) \quad \forall k, \quad 0 \leq \theta_1, \theta_2 < \frac{2\pi}{N}$$

which implies that  $\mathbf{q}(\theta)$  is piecewise constant. We need only consider  $\tilde{\mathbf{q}} \triangleq \tilde{\mathbf{q}}(0)$  since it contains one sample from each constant segment of  $\mathbf{q}(\theta)$ . Henceforth, whenever  $\theta = 0$ , the  $\theta$ -dependence will be omitted with no risk of confusion.

We first consider the system corresponding to  $M = 2$ . In this case (8) reduces to a system of  $N$  linear equations. Specifically, let  $[\mathbf{p}(\theta)]_k = p(1, \theta - (k-1)\omega_o)$ . We have

$$\mathbf{p}(\theta) = \mathbf{D}(\theta)\mathbf{p}(\theta) + \mathbf{b}(\theta) \quad (10)$$

where  $[\mathbf{b}(\theta)]_k = \alpha_1(\theta - (k-1)\omega_o)/(2\pi)$ ,

$$[\mathbf{D}(\theta)]_{k,j} = \begin{cases} d(\theta - k\omega_o) & \text{if } k = j + 1 \bmod N \\ 0 & \text{otherwise,} \end{cases}$$

and  $d(\theta) = 1 - \sum_i \alpha_i(\theta)$ , where  $\alpha_1(\theta) = \Pr[w[n] \geq 1 + g(\theta)]$ . Using (5) and since  $w[n]$  is i.i.d. and evenly-distributed, we obtain that  $\alpha_i(\theta)$  is two-valued for each  $i$  and that  $d(\theta) = d(0) = \lambda_2$ . From this and (10), we obtain (6) for  $M = 2$ .

The proof for  $M > 2$  is similar. Here we assume for  $p_i[n]$  the form specified by (6) and show how the  $k_{i,j}$ 's are determined. We can readily show that  $\mathbf{k}_j = c_j \mathbf{v}_j$ , where  $\mathbf{v}_j$  is the eigenvector associated with  $\lambda_j$ , and the  $c_j$ 's are

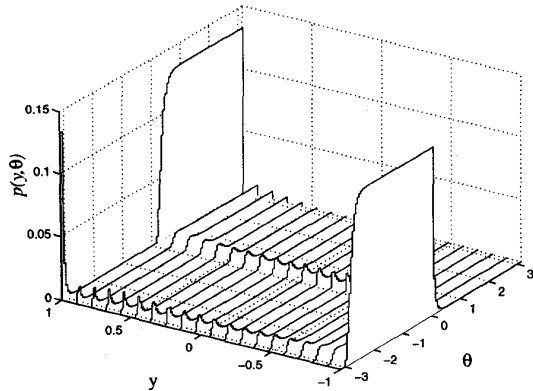


Figure 1: Output invariant density for a quantizer map ( $M = 20$ ) driven by a noisy square wave.

scalar constants, by substituting the expression in (6) for  $\tilde{\mathbf{q}}(\theta)$  in (8) and setting the coefficients of  $\lambda_j$  equal for each  $j$ . The condition  $\mathbf{1}^T \mathbf{k}_1 = 1/(2\pi)$  ( $\mathbf{1}$  is a column vector of 1's) gives  $c_1 = (2\pi \mathbf{1}^T \mathbf{v}_1)^{-1}$ . The boundary condition at  $n = 0$  and the symmetry condition  $p(Y_i, \theta) = p(-Y_i, \theta - \pi)$  provide the remaining  $c_j$ 's. Specifically, let  $\tilde{\mathbf{B}} = \mathbf{V} \mathbf{A} \mathbf{V}^{-1}$  denote the eigenvalue decomposition of  $\tilde{\mathbf{B}}$ , and  $\tilde{\mathbf{I}}$  denote the  $M \times M$  permutation matrix with ones on the main anti-diagonal. Then, the vector  $\{\mathbf{c} : [\mathbf{c}]_j = c_j\}$  is the eigenvector of the matrix  $\mathbf{V} - \tilde{\mathbf{I}} \mathbf{V} \mathbf{A}^{-N/2}$  associated with its zero eigenvalue. Eqn. (6) can be readily verified by substituting for each entry of  $\tilde{\mathbf{q}}$  in (8) the appropriate expression from (6). ■

Shown in Fig. 1 is the invariant density of the output of a quantizer map with  $M = 20$  and  $\gamma = 1$ , driven by a square wave of amplitude  $A = 0.4$  and period  $N = 100$ , and where  $w[n]$  is white normally-distributed with  $\sigma_w^2 = 0.16$ . Note that the distribution for any particular value of  $y = Y_t$  as a function of  $\theta$  is piecewise constant, and takes one of two values, except for the short transient in the transition regions.

$M$ -level quantizer maps can be used to approximate a wide class of bistable and multi-stable systems. As an example, we consider the system (4), where  $F(\cdot)$  is a quantizer approximating the following map

$$\tilde{F}_\epsilon(y) = (1 + \epsilon)y - \epsilon y^3$$

with  $\epsilon = 0.08$ . The system is driven by a square wave with  $A = 0.03$  and  $N = 1000$ , in white Gaussian noise of variance  $\sigma_w^2 = 0.01$ . The solid and dotted curves in Fig. 2 correspond to the limiting state densities as a function of  $y$ , for  $s[n] = A$  and  $s[n] = -A$ , respectively. The map  $\tilde{F}_\epsilon(\cdot)$  has two stable fixed points. In the absence of noise,  $s[n]$  cannot induce a transition from one stable fixed point to the other. However, as Fig. 2 reveals, noise causes the output of the map to follow the drive with very high probability. This is the essence of stochastic resonance.

The resulting density function was obtained by approximating  $\tilde{F}_\epsilon(\cdot)$  with uniform quantizers of successively larger values of  $M$  until sufficient convergence was achieved. The map  $\tilde{F}_\epsilon(\cdot)$  arises by uniform sampling of  $y(t)$  and approximating the derivative with a forward difference, of the

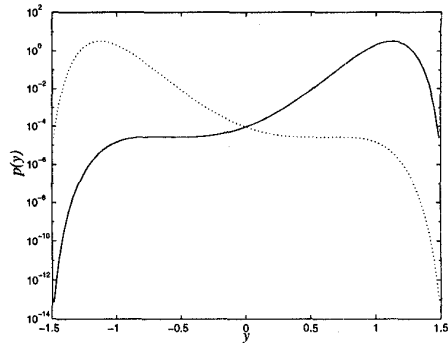


Figure 2: Limiting output state densities for a bistable map driven by a noisy square wave.

continuous-time bistable system  $\dot{y} = y - y^3$ , a widely studied system exhibiting stochastic resonance.

#### 4. OUTPUT SNR FOR QUANTIZER MAPS

The notion of input SNR is well-defined for a period- $N$  square wave signal in i.i.d. finite-variance noise. For normally-distributed noise, such a notion of SNR naturally relates to the probability of detection for an optimal detector. One of the commonly used notions of output SNR of a map in the context of a periodic drive is based on comparison of the signal power and noise power spectral density at the fundamental frequency  $\omega_o = 2\pi/N$ . We obtain the output SNR for the quantizer maps for  $M = 2$  through the autocorrelation function of the output by separating it into signal and noise components.

**Proposition 2** Consider the map (4) where  $F(\cdot)$  is given by (2) for  $M = 2$ ,  $g(\cdot)$  is given by (5), and  $w[n]$  is an i.i.d. evenly-distributed noise sequence. The phase-averaged autocorrelation of the output  $y[n]$  can be decomposed into periodic (signal) and aperiodic (noise) components, i.e.,

$$R[n] = R_S[n] + R_N[n].$$

The component  $R_S[n]$  is  $N$ -periodic, specifically

$$R_S[n] = C_S \left(1 - \frac{4}{N} |n|\right) + \frac{C_1}{N} (\lambda_2^{|n|} - \lambda_2^{N/2 - |n|}) \quad (11)$$

for  $-N/2 < n < N/2$ , where

$$C_S = \frac{(\beta_l - \beta_s)^2}{(1 - \lambda_2)^2}, \quad C_1 = 2C_S \frac{2(\lambda_2 + 2)}{(1 - \lambda_2^2)(1 + \lambda_2^{N/2})}.$$

The aperiodic component is given by  $R_N[n] = C_N \lambda_2^{|n|}$ , where

$$C_N = 4 \left(1 - \frac{\beta_l^2 + \beta_s^2}{(1 - \lambda_2)^2} + 2 \frac{(\beta_l - \beta_s)^2 (1 - \lambda_2^{N/2})}{(1 - \lambda_2)^3 (1 + \lambda_2)(1 - \lambda_2^{N/2})}\right).$$

We note that for large  $N$  (i.e.,  $N \gg 4(\lambda_2 + 2)/(1 - \lambda_2^2)$ ), the second term in (11) is negligible, and the resulting  $R_S[n]$  approximates the autocorrelation function of a period- $N$  square wave of amplitude  $\sqrt{C_S}$ .

*Proof:* Using an approach analogous to the one used to obtain (8), we can write an equation for the autocorrelation

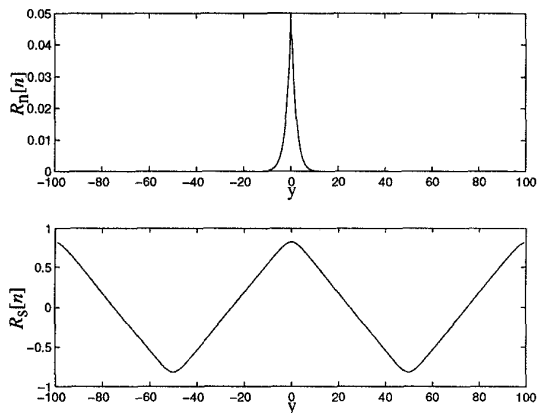


Figure 3: Noise and signal components of the phase-averaged autocorrelation function for the output of a quantizer map ( $M = 20$ ) driven by a noisy square wave.

function for the systems corresponding to  $M = 2$  given a particular value of  $\theta$ , *i.e.*,

$$\tilde{\mathbf{r}}[n; \theta] = \tilde{\mathbf{T}}^n(\theta) \tilde{\mathbf{q}}(\theta)$$

where  $\tilde{\mathbf{T}}(\theta)$  is given by the right hand side of (9), with  $\mathbf{B}(\theta)$  replaced by  $\mathbf{T}(\theta)$ , where  $[\mathbf{T}(\theta)]_{i,j} = [\mathbf{B}(\theta)]_{i,j} Y_i Y_j$ . Since  $\tilde{\mathbf{r}}[n; \theta]$  is constant over segments of length  $2\pi/N$ ,

$$R[n] = \frac{2\pi}{N} \mathbf{1}^T \tilde{\mathbf{r}}[n; 0] = \frac{2\pi}{N} \mathbf{1}^T \tilde{\mathbf{T}}^n \tilde{\mathbf{q}}. \quad (12)$$

The form of  $\tilde{\mathbf{T}}$  implies that it has  $N$  eigenvalues uniformly spaced on the unit circle, while the rest are uniformly spaced on a circle of radius  $\lambda_2 < 1$ . The periodic component of  $R[n]$  is associated with the unit-magnitude eigenvalues. It is convenient to separate the  $\lambda_1$ - from the  $\lambda_2$ -magnitude eigenvalues by setting  $\tilde{\mathbf{T}} = \tilde{\mathbf{T}}_s + \tilde{\mathbf{T}}_n$ , where  $\tilde{\mathbf{T}}_s, \tilde{\mathbf{T}}_n$  are obtained from  $\tilde{\mathbf{T}}$ , by setting to zero all  $\lambda_2$ -,  $\lambda_1$ -magnitude eigenvalues, respectively. Then,  $R[n] = R_s[n] + R_n[n]$ , where  $R_s[n]$  ( $R_n[n]$ ) are given by substituting  $\tilde{\mathbf{T}}_s$  ( $\tilde{\mathbf{T}}_n$ ) for  $\tilde{\mathbf{T}}$  in (12). Since  $\tilde{\mathbf{T}}_s^n$  is  $N$ -periodic,  $R_s[n]$  is also. It is convenient to view  $\tilde{\mathbf{T}}_s^n$  as a matrix composed of  $2 \times 2$  blocks; the  $k$ -th block row has all zero blocks except the  $(n+k) \bmod N$ -th block, whose columns are both equal to  $\mathbf{q}(-2\pi k/N)$ . Using this property and (6) for  $M = 2$ , we obtain (11). The aperiodic component,  $R_n[n]$ , can be similarly computed by considering the eigenvalues of magnitude  $\lambda_2$ . ■

It is worth noting that via this result we can readily derive certain higher-order statistics for the map corresponding to  $M = 2$ . In particular, since  $y^m[n] = 1$  for even  $m$ , the higher-order statistics  $E[y^m[n]y^l[k]]$  are given in terms of the first- and second-order statistics of the process.

Via a generalization of Proposition 2, we may also obtain the autocorrelation function for  $M > 2$  [8]. In Fig. 3 we plot the noise and signal components of the phase-averaged autocorrelation function for the system corresponding to  $M = 20$ ,  $\gamma = 1$ , driven by a square-wave of amplitude  $A = 0.4$  and period  $N = 100$ , in white Gaussian noise of variance  $\sigma_w^2 = 0.16$ . Note that, as  $N$  becomes large, the coherent component  $R_s[n]$  approaches the autocorrelation function of a square wave, *i.e.*, a triangular wave.

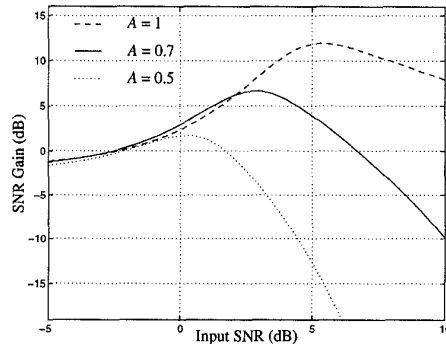


Figure 4: SNR gain based on power spectra for a quantizer map ( $M = 2$ ).

A natural notion of signal-to-noise ratio gain for these one-dimensional systems is readily suggested by the above decomposition of the output autocorrelation function. Using this measure, in Fig. 4 we plot the SNR gain for the system corresponding to  $M = 2$  driven by a square-wave of period  $N = 1000$  for various amplitude levels as a function of input SNR. In particular, input and output SNR levels are computed as ratios of the respective coherent to noise power spectral density at the fundamental frequency. It is interesting to note that the SNR gain is positive for a fairly wide input SNR range. The gain reduction at high input SNR can be easily justified; the noise level is not high enough to push the output over the short transition barrier between the two stable points. The relationship of this SNR gain to other quantities of interest that arise naturally in signal processing applications is explored in [8].

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