STATISTICAL PROPERTIES OF ONE-DIMENSIONAL CHAOTIC SIGNALS

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ABSTRACT

Signals arising out of nonlinear dynamical systems are compelling models for a wide range of phenomena. We develop several properties of signals obtained from Markov maps, an important family of such systems, and present analytical techniques for computing their statistics. Among other results, we demonstrate that all Markov maps produce signals with rational spectra, and can therefore be viewed as "chaotic ARMA processes." Finally, we demonstrate how Markov maps can approximate to arbitrary precision any of a broad class of chaotic maps and their statistics.

1. INTRODUCTION

Recently, there has been considerable interest in developing useful classes of signals out of nonlinear dynamics and chaos theory. In this paper, we consider discrete-time signals $x[n]$ generated by chaotic systems with a single state variable by applying the recursion

$$x[n] = f(x[n-1]) \tag{1}$$

to some initial condition $x[0]$, where the $f(\cdot)$ is a nonlinear transformation that maps scalars to scalars. We restrict our attention to the class of such systems governed by piecewise-smooth dynamics, which have been proposed as models not only for a variety of physical phenomena but also for engineering systems ranging from nonlinear oscillators [1] to switching power converters [2].

Many of these applications require knowledge of the time-average properties—such as power spectra and higher-order moments—of time-series generated according to (1). In developing our results, we will exploit powerful results from ergodic theory that allow us to equate the time-average properties of such deterministic signals with the ensemble-average properties of the class of stationary stochastic processes generated according to the dynamics (1) with an initial condition $x[0]$ chosen from an appropriate probability distribution.

To develop the properties of the corresponding stochastic process, we begin by using $p_0(\cdot)$ to denote the probability density function of the initial condition $x[0]$, and more generally $p_n(\cdot)$ to denote the corresponding density of $x[n]$. When we restrict our attention to maps $f(\cdot)$, for example, that consist of a finite number of piecewise differentiable segments with no points of zero-slope, a linear operator $P_f : L_1 \rightarrow L_1$ may be defined such that

$$p_n(\cdot) = P_f\{p_{n-1}(\cdot)\}. \tag{2}$$

This operator, referred to as the Frobenius–Perron (FP) operator [3], thus describes the time-evolution of the density for the particular map.

Although in general, the densities at distinct times n will differ, there can exist certain choices of $p_0(\cdot)$ such that the pdf of subsequent iterates does not change, i.e.,

$$p_0(\cdot) = p_1(\cdot) = \cdots = p_n(\cdot) \stackrel{\triangle}{=} p(\cdot). \tag{3}$$

The density $p(\cdot)$ is therefore referred to as an invariant density of the map f, and is a fixed point of the FP operator, i.e.,

$$p(\cdot) = P_f\{p(\cdot)\}. \tag{4}$$

When p_0 is chosen to be an invariant density, it may be shown that the resulting stochastic process is stationary and—subject to certain constraints on f—ergodic. In this case, Birkhoff's ergodic theorem [4] can be used to establish that time-averages are equivalent to ensemble-averages for almost all sample waveforms.

In this paper, we concentrate on correlation statistics of the form

$$R_{f;h_0,h_1,\ldots,h_r}[k_1,\ldots,k_r] = \lim_{L\to\infty} \frac{1}{L}\sum_{n=0}^{L-1} h_0(x[n])h_1(x[n+k_1])\cdots h_r(x[n+k_r]), \tag{5}$$

where $x[n]$ is a time-series generated by (1), the $h_i(\cdot)$ are well-behaved functions and the k_i are nonnegative integers. These statistics are defined generally enough to include, e.g., the autocorrelation and all higher-order moments of the time-series that are of interest in a broad range of chaotic data analysis and synthesis problems.

Using $f^n(\cdot)$ and $P_f^n\{\cdot\}$ to denote the respective n-fold compositions of $f(\cdot)$ and $P_f\{\cdot\}$ with themselves, and defining $\Delta_i = k_i - k_{i-1}$ with $k_0 = 0$, the correlation statistic (5) can be expressed in the form

$$R_{f;h_0,h_1,\ldots,h_r}[k_1,\ldots,k_r] = \lim_{L\to\infty} \frac{1}{L}\sum_{n=0}^{L-1}\prod_{i=0}^{r} h_i(f^{k_i}(x[n]))$$

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$$= \int p(x) \prod_{i=0}^{r} h_i(f^{k_i}(x))\, dx$$

$$= \int h_r(x)\, P_f^{\Delta_r}\{h_{r-1}(x) \cdots P_f^{\Delta_2}\{h_1(x)\, P_f^{\Delta_1}\{h_0(x)\, p(x)\}\} \cdots\}\, dx, \qquad (6)$$

where we have used, in turn, (1), Birkhoff's ergodic theorem, and repeated application of the identity

$$\int h_1(x) P_f\{h_2(x)\} dx = \int h_1(f(x)) h_2(x) dx$$

valid for integrable $h_i(\cdot)$. The form (6), which makes explicit the dependency of correlation statistics on both the FP operator and the associated invariant density, will be particularly useful in computing these statistics for the class of so-called Markov maps, as we develop next.

2. PIECEWISE LINEAR MARKOV MAPS

One rich class of one-dimensional chaotic systems are the eventually-expanding piecewise-linear Markov maps. For convenience, we restrict our attention to Markov maps of the unit interval, which are defined as follows.

Definition 1 *A map* $f : [0,1] \to [0,1]$ *is an* eventually-expanding, piecewise-linear, Markov map *when*

1. *There is a set of* partition points $0 = a_0 < a_1 < \cdots < a_N = 1$ *such that restricted to each of the intervals* $\mathcal{V}_i = (a_{i-1}, a_i)$, *the map* f *is affine.*
2. *For each* i, $f(a_i) = a_j$ *for some* j.
3. *There is an integer* $k > 0$ *such that* $\inf_{x \in [0,1]} |\frac{d}{dx} f^k(x)| > 1$ *for all* x.

Among other important properties, all Markov maps have invariant densities and are ergodic under readily verifiable conditions [5]. Interestingly, and rather remarkably, suitably quantized outputs of Markov map dynamics are also Markov chains. However, of particular importance for our purposes is that Markov maps have statistics that can be determined in closed form and used to approximate, to arbitrary accuracy, statistics of a much larger class of chaotic systems.

From Definition 1, we may express a Markov map in the form

$$f(x) = \sum_{i=1}^{N} (s_i\, x + b_i) \chi_i(x), \qquad (7)$$

where $|s_i| > 0$ and where $\chi_i(x)$ is the indicator function taking the value 1 when $x \in \mathcal{V}_i$ and 0 otherwise.

3. STATISTICS OF MARKOV MAPS

In this section, we develop a compact representation of the statistics of Markov maps by exploiting a key result: a Markov map's FP operator, which can be expressed in the form (see e.g. [6])

$$P_f\{h(x)\} = \sum_{i=1}^{N} h((x - b_i)/s_i)\, \chi_{f(\mathcal{V}_i)}(x)/|s_i|. \qquad (8)$$

has finite-dimension when restricted to certain subspaces.

Consider, e.g., piecewise polynomial $h(\cdot)$ of the form

$$h(x) = \sum_{i=1}^{N} \sum_{j=0}^{k} a_{ij} x^j \chi_i(x) \triangleq \sum_{i=1}^{N(k+1)} h_i \theta_i(x), \qquad (9)$$

where the a_{ij} are arbitrary scalars, and where

$$\{\theta_1, \ldots, \theta_{N(k+1)}\} \triangleq \{\chi_1, \ldots, \chi_N, x\chi_1, \ldots, x\chi_N, \ldots, x^k\chi_1, \ldots, x^k\chi_N\} \qquad (10)$$

We denote by $\mathcal{P}_k$ the $N(k+1)$-dimensional space spanned by $\{\theta_i\}$. Thus, each piecewise polynomial in $\mathcal{P}_k$ is uniquely specified by the $N(k+1)$-tuple $\mathbf{h} = [h_1, \ldots, h_{N(k+1)}]^T$ which we refer to as the coordinate vector of $h(x)$.

For these piecewise polynomial functions, substituting (9) into (8) and exploiting Definition 1 yields, after some straightforward manipulation,

$$P_f\{h(x)\} = \sum_{i=1}^{N} \sum_{j=0}^{k} a_{ij} \left(\frac{x - b_i}{s_i}\right)^j \frac{1}{|s_i|} \sum_{l \in \mathcal{I}_i} \chi_l(x). \qquad (11)$$

where $\mathcal{I}_i$ denotes the set of indices of partition elements in the image of $\mathcal{V}_i$, i.e., $f(\mathcal{V}_i) = \cup_{j \in \mathcal{I}_i} \mathcal{V}_j$. From (11) we therefore obtain that the image of $h(\cdot)$ is also piecewise-polynomial of degree k.

Since the operator $P_f\{\cdot\}$ maps $\mathcal{P}_k$ to itself, its restriction to $\mathcal{P}_k$ can be represented by a square $N(k+1)$ dimensional matrix, which we denote by $\mathbf{P}_k$. In particular, this matrix describes how the coefficients of expansions in terms of the basis (10) map under the FP operator.

Using (11) with the binomial theorem, we get that $\mathbf{P}_k$ takes the block upper-triangular form

$$\mathbf{P}_k = \begin{bmatrix} \mathbf{P}_{00} & \mathbf{P}_{01} & \cdots & \cdots & \mathbf{P}_{0k} \\ 0 & \mathbf{P}_{11} & \mathbf{P}_{12} & \cdots & \mathbf{P}_{1k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \mathbf{P}_{kk} \end{bmatrix}, \qquad (12)$$

where each nonzero $N \times N$ block is of the form

$$\mathbf{P}_{ij} = \binom{j}{i} \mathbf{P}_0 \mathbf{B}^{j-i} \mathbf{S}^j \qquad j \geq i. \qquad (13)$$

The $N \times N$ matrices $\mathbf{B}$ and $\mathbf{S}$ are diagonal with elements $B_{ii} = -b_i$ and $S_{ii} = 1/s_i$, respectively, while $\mathbf{P}_0 = \mathbf{P}_{00}$ is the $N \times N$ matrix with elements

$$[\mathbf{P}_0]_{ij} = \begin{cases} 1/|s_j| & \text{if } i \in \mathcal{I}_j \\ 0 & \text{otherwise.} \end{cases} \qquad (14)$$

The invariant densities of a Markov map can be obtained by solving a matrix eigenvalue problem. In particular, the invariant densities of a Markov map $f(\cdot)$ are elements of $\mathcal{P}_0$, i.e., piecewise-constant, and can therefore be obtained from the solution to

$$\mathbf{P}_0 \mathbf{p} = \mathbf{p}, \qquad (15)$$

where $\mathbf{p} = [p_1, \ldots, p_N]^T$ is the coordinate vector of the invariant density $p(x)$. From (15) we see that this coordinate vector is the eigenvector of $\mathbf{P}_0$ corresponding to the eigenvalue 1. That such an eigenvector/eigenvalue pair always exists has been verified by Friedman and Boyarsky [7].

More generally we now consider the computation of correlation statistics (5) when $h_i(\cdot) \in \mathcal{P}_k$ for each i. Exploiting our basis expansions we can rewrite (6) in the form

$$R_{f;h_0,h_1,\ldots,h_r}[k_1,\ldots,k_r] = \int g_1(x)\, g_2(x)\, dx = \mathbf{g}_1^T \mathbf{M} \mathbf{g}_2 \qquad (16)$$

where

$$[\mathbf{M}]_{ij} = \int \theta_i(x)\, \theta_j(x)\, dx, \qquad (17)$$

and where $\mathbf{g}_1 = \mathbf{h}_r$ and[1]

$$\mathbf{g}_2 = \mathbf{P}^{\Delta_r}(\mathbf{h}_{r-1} \odot \cdots \odot \mathbf{P}^{\Delta_2}(\mathbf{h}_2 \odot \mathbf{P}^{\Delta_1}(\mathbf{h}_1 \odot \mathbf{p}))\cdots)$$

are the respective coordinate vectors of the functions $g_1(x) = h_r(x)$ and

$$g_2(x) = P_f^{\Delta_r}\{h_{r-1}(x) \cdots P_f^{\Delta_2}\{h_1(x)\, P_f^{\Delta_1}\{h_0(x)\, p(x)\}\} \cdots\}.$$

The compact representation (16) can be used, e.g., to determine the power spectrum associated with a Markov map. In particular, with $h_1(x) = x$ and $h_2(x) = xp(x)$ we obtain the autocorrelation sequence in the form [6]

$$R_{f;xx}[k] = \int x f^k(x)\, p(x)\, dx = \mathbf{h}_1^T \mathbf{M} \mathbf{P}_1^{|k|} \mathbf{h}_2$$

which, after taking the Fourier transform, yields

$$S_{xx}(e^{j\omega}) = \mathbf{h}_1^T \mathbf{M} \left(\sum_{n=-\infty}^{\infty} \mathbf{P}_1^{|n|} e^{-j\omega n} \right) \mathbf{h}_2. \qquad (18)$$

Unit-magnitude eigenvalues of the FP matrix $\mathbf{P}_1$ lead to impulses in the Fourier transform (18), and there typically exists at least one such eigenvalue due to the nonzero mean of the time-series. To eliminate consideration of such spectral lines in our analysis, we use a Jordan-form decomposition to obtain

$$\mathbf{\Gamma} = \mathbf{E}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{J} \end{bmatrix} \mathbf{E}$$

where $\mathbf{E}$ is the matrix of generalized eigenvectors of $\mathbf{P}_1$, and $\mathbf{J}$ is the matrix of associated eigenvalues having magnitude strictly less than one.

We can rewrite the non-impulsive part of (18) using this decomposition to obtain

$$S_{xx}(e^{j\omega}) = \mathbf{h}_1^T \mathbf{M}(\mathbf{I} - \mathbf{\Gamma} e^{-j\omega})^{-1}(\mathbf{I} - \mathbf{\Gamma}^2)(\mathbf{I} - \mathbf{\Gamma} e^{j\omega})^{-1} \mathbf{h}_2.$$

It is immediately apparent that $S_{xx}(e^{j\omega})$ is a rational function. Furthermore, its poles correspond to the eigenvalues of the matrix $\mathbf{\Gamma}$, while the zeros depend on the vectors $\mathbf{h}_1$ $\mathbf{h}_2$ and the matrix $\mathbf{M}$.

[1] $\mathbf{P}$ is an appropriately dimensioned matrix representation of the FP operator, and for two coordinate vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, the notation $\mathbf{u}_1 \odot \mathbf{u}_2$ denotes the coordinate vector for the corresponding product of piecewise polynomials $u_1(x)\, u_2(x)$.

4. MODELING WITH MARKOV MAPS

A much larger class of chaotic signals are obtained from *eventually-expanding* maps, which are defined as follows.

Definition 2 *A nonsingular map* $f : [0,1] \to [0,1]$ *is called* eventually-expanding *if*

1. *There is a set of partition points* $0 = a_0 < a_1 < \cdots a_N = 1$ *such that restricted to each of the intervals* $\mathcal{V}_i = (a_{i-1}, a_i)$, *the map* $f(\cdot)$ *is monotonic, continuous and differentiable.*
2. *The function* $\frac{1}{|f'(x)|}$ *has bounded variation.*
3. *There is a number* $\lambda > 1$ *and an integer* m *such that* $|\frac{d}{dx} f^m(x)| \geq \lambda$ *wherever the derivative exists.*

The class of eventually-expanding maps, includes among others, non-piecewise-affine maps and maps with unbounded slope. All eventually-expanding maps have invariant densities [8] and, more importantly, can be modelled arbitrarily well by Markov maps.

To see this, let us consider a sequence $\hat{f}_i(\cdot)$ of Markov maps and examine conditions under which the statistics of $\hat{f}_i(\cdot)$ converge to those of a given eventually-expanding map $f(\cdot)$. This important mode of convergence, which we call *statistical convergence*, is defined as follows.

Definition 3 *Let* $f(\cdot)$ *be an eventually-expanding map with a unique invariant density* $p(\cdot)$. *A sequence of maps* $\{\hat{f}_i(\cdot)\}$ statistically converges *to* $f(\cdot)$ *if each* $\hat{f}_i(\cdot)$ *has a unique invariant density* $p_i(\cdot)$ *and*

$$R_{\hat{f}_n,h_0,\ldots,h_l}[k_1,\ldots,k_r] \to R_{f,h_0,\ldots,h_l}[k_1,\ldots,k_r]$$

for any continuous $h_i(\cdot)$ *and all finite* k_i *and finite* r.

A sequence of Markov maps that statistically converges to a given eventually-expanding map $f(\cdot)$ can be constructed in a computationally straightforward manner. To begin, we denote by $\mathcal{Q}$ the set of partition points of $f(\cdot)$ and define the sequence of increasingly fine partitions $\mathcal{Q}_i = \mathcal{Q}_{i-1} \cup f^{-1}(\mathcal{Q}_{i-1})$. Each $\hat{f}_i(\cdot)$ is defined by specifying its value at the partition points $\mathcal{Q}_i$.

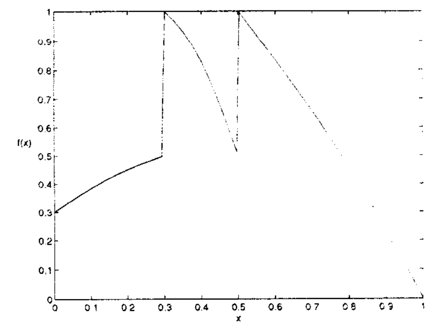
1. For each partition point $q \in \mathcal{Q}_i$ such that $f(\cdot)$ is increasing at q^+, define $\hat{f}_i(q^+)$ by
$$\hat{f}_i(q^+) = \max\{v \in \mathcal{Q}_i | v \leq f(q^+)\}.$$
2. For each partition point $q \in \mathcal{Q}_i$ such that $f(\cdot)$ is decreasing at q^+, define $\hat{f}_i(q^+)$ by
$$\hat{f}_i(q^+) = \min\{v \in \mathcal{Q}_i | v \geq f(q^+)\}.$$
3. For each partition point $q \in \mathcal{Q}_i$ such that $f(\cdot)$ is increasing at q^-, define $\hat{f}_i(q^-)$ by
$$\hat{f}_i(q^-) = \min\{v \in \mathcal{Q}_i | v \geq f(q^-)\}.$$
4. For each partition point $q \in \mathcal{Q}_i$ such that $f(\cdot)$ is decreasing at q^-, define $\hat{f}_i(q^-)$ by
$$\hat{f}_i(q^-) = \max\{v \in \mathcal{Q}_i | v \leq f(q^-)\}.$$

The map $\hat{f}_i(\cdot)$ is defined at all other points by linear interpolation. It is straightforward to verify that each $\hat{f}_i(\cdot)$ is an eventually-expanding, piecewise-linear Markov map [9].

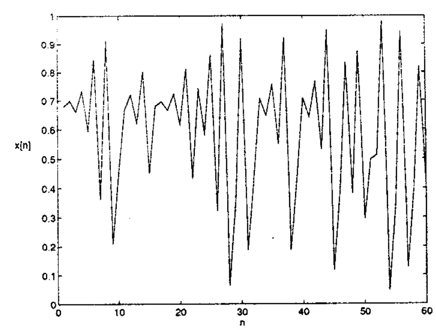
It can be shown [6] that the sequence of Markov approximations to f described above statistically converges to f. Hence, for sufficiently large i, the statistics of $\hat{f}_i(\cdot)$ are close to those of $f(\cdot)$. This has the practical consequence that correlation statistics of an eventually-expanding map $f(\cdot)$ can be approximated by first determining a Markov map $\hat{f}_k(\cdot)$ that is a good approximation to f, then finding the statistics of Markov map using the techniques described in the previous section.

This approach can be used, in particular, to approximate the power spectrum associated with an eventually-expanding map. Consider, for example, the map shown in Figure 1 (a). As illustrated in Figure 1 (b), a time-series generated by this map alternates irregularly between periods of exponential growth in amplitude and periods of rapid decay, which results in a pronounced peak in its power spectrum. Figure 1 (c) shows an empirically computed power spectrum along with two approximate spectra. The dashed line and solid line correspond to approximate spectra computed analytically using the Markov maps $\hat{f}_1(\cdot)$ and $\hat{f}_3(\cdot)$, while the dash-dotted line corresponds to an estimate of the power spectrum associated with $f(\cdot)$ that was determined by applying periodogram averaging to a time-series generated by $f(\cdot)$.

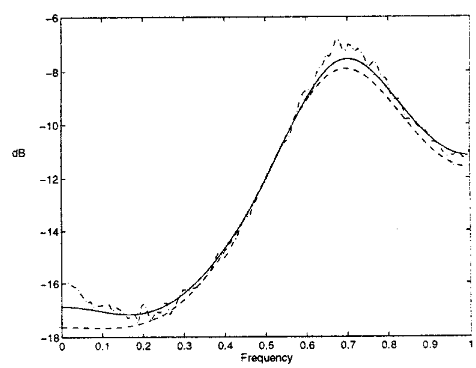
A more extensive development of the results of this paper is contained in [6].



(a) An Example of a Non-Affine Eventually-Expanding Map



(b) A Segment of a Typical Time-Series Generated by $f(x)$



(c) Comparison of Approximate and Empirical Power Spectra

Figure 1: Example Map

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