# EFFECTS OF CONVOLUTION ON CHAOTIC SIGNALS

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## ABSTRACT

Because chaotic signals are potentially useful both in describing physical phenomena and in engineering applications, signal processing algorithms exploiting their unique characteristics are of interest. In this paper, we consider issues pertaining to processing signals in convolutional distortion. Specifically, we discuss the effects of convolutional distortion on two parameters commonly used in the description of chaotic signals – the Lyapunov exponents and the fractal dimension of the attractor. In addition, we present a blind deconvolution technique based on minimizing a nonlinear prediction error for data generated by one dimensional chaotic maps.

## 1. INTRODUCTION

Chaotic signals are of increasing interest in science and engineering because they model a wide range of physical phenomena and contain a considerable amount of inherent structure. Signals which are well modelled as chaotic have been observed in physical phenomena ranging from turbulence in fluid flow to radar clutter [4]. Also, because of the deterministic structure of chaotic signals, they are potentially applicable in areas such as communications systems [6]. In these contexts and others, we often have access only to distorted versions of the chaotic signals of interest and would like to make inferences concerning the original signal. This paper is concerned with chaotic signals distorted by convolution. This is an important distortion mechanism to consider because a wide variety of phenomena affecting applications –e.g. propagation channel effects and signal smearing due to sensor motion– are well modelled by convolution.

In this paper, we examine the effects of convolutional distortion on two sets of parameters of chaotic signals – the Lyapunov exponents and fractal dimension of the attractor. These quantities are defined in section 2. Their behavior under convolution is discussed in Section 3. Finally, in section 4 we present a deconvolution technique for signals generated from one dimensional chaotic difference equations.

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## 2. BACKGROUND

We confine our attention to signals generated by an $N$ dimensional difference equation of the form

$$\begin{aligned} \mathbf{x}[n+1] &= \mathbf{F}(\mathbf{x}[n]) \\ y[n] &= G(\mathbf{x}[n]) \end{aligned} \tag{1}$$

where $\mathbf{x}[n]$ is the state at time $n$ and $y[n]$ is a scalar observation of the state at time $n$. We assume that $G$ is Lipschitz and that $\mathbf{F}$ is a Lipschitz[1] diffeomorphism ($LD$) which displays dissipative chaotic behavior. The dissipative nature of $\mathbf{F}$ implies that state trajectories eventually converge to an attractor which has zero volume when viewed in the state space. Also, since the system is chaotic it displays sensitive dependence on initial conditions, i.e. trajectories generated from initial conditions which are arbitrarily close in state space will eventually diverge exponentially from one another. This local instability implies that the scalar observation $y[n]$ will be a broadband signal. Of course, this divergence cannot continue indefinitely since the trajectories must remain bounded. Eventually widely separated trajectories must fold back toward each other creating a fractal structure.

Two signature quantities of chaotic systems and the signals they generate are Lyapunov exponents and the fractal dimension of the underlying attractor. Lyapunov exponents describe the local sensitivity to initial conditions, while the fractal dimension quantifies the notion of the size of the limiting trajectory of the system. These notions are defined more precisely below. A more detailed treatment of these issues is available in several references, e.g. [1].

In order to define the Lyapunov exponents we make use of the notion of stable and unstable directions. At each point on the attractor there exists a set of stable and unstable directions $\{u_k\}$ Initial conditions differing by a small perturbation equal to one of these directions generate state sequences which exponentially converge or diverge at a rate determined by the Lyapunov exponent. These exponents are defined through a linear approximation to the distance between trajectories. Specifically, the Lyapunov exponents satisfy the relation

$$\lim \frac{1}{n} \log \|\nabla(\mathbf{F}^n(x))u_i\| = \lambda_i. \tag{2}$$

Positive Lyapuov exponents correspond to expanding directions while negative exponents correspond to contracting

[1] A function $\mathbf{F}$ is Lipschitz if $\|\mathbf{F}(x) - \mathbf{F}(y)\| < K\|x - y\|$ for all $\mathbf{x}$ and $\mathbf{y}$ in its domain and for some finite constant $K$. This condition essentially restricts the maximum growth rate of a function to be finite.



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directions. Under mild conditions it can be shown that the $\lambda_i$ do not depend on the value of $x$ when $x$ is a point of the attractor.

Another quantity commonly used to characterize chaotic systems is the fractal dimension of its attractor. Its importance as a signature quantity can be seen by noting that all $m$ dimensional dissipative chaotic systems of the form (1) will have attractors with zero volume in $\mathbb{R}^m$. The fractal dimension is a way of comparing the size of these zero volume sets. Of the many notions of fractal dimension, perhaps the easiest to define is the capacity. Given a set in $\mathbb{R}^m$, let $N(\epsilon)$ be the minimum number of $m$-dimensional balls of radius $\epsilon$ needed to cover the set. The capacity dimension of the set quantifies the rate at which $N(\epsilon)$ grows as $\epsilon$ goes to zero and is defined as

$$D_c = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{-\log \epsilon}.$$

It is straightforward to verify that for simple sets in $\mathbb{R}^m$ such as line segments and cubes , the value of the capacity corresponds to our intuitive notion of the dimension.

Under our assumptions, a theorem of Takens [10] states that for most systems of the form (1), vectors constructed from the scalar observation $y[n]$ of a chaotic state sequence using the so-called time delay reconstruction

$$\mathbf{Y}[n] = [y[n], y[n-1], \cdots, y[n-\tilde{N}+1]]$$

are equivalent (to within a nonlinear coordinate change by a $LD$) to the original state vectors $\mathbf{x}[n]$ for sufficiently large $\tilde{N}$. Further, it can be shown that that the capacity dimension and Lyapunov exponents are invariant to Lipschitz diffeomorphisms. These two facts together imply that the dimension and exponents of the original system may be estimated from an observed scalar time series. This will be of use in the next section where we examine the effects of filtering on chaotic signals. For a review of techniques for estimating these quantities from data see [1].

## 3. EFFECTS OF CONVOLUTION

Consider the situation in which we have access only to a distorted version $z[n]$ of the signal $y[n]$. The relationship between the distorted and the desired signal is given by

$$z[n] = \sum_{i=0}^{\infty} h[i] y[n-i]. \qquad (3)$$

where $h[n]$ is the impulse response of a stable causal LTI system with a rational transfer function.

In order to determine the effects of filtering on Lyapunov exponents, we represent the time series $z[n]$ as the scalar observation of a composite system of the original nonlinear dynamics and the filtering dynamics :

$$\begin{aligned} \mathbf{x}[n+1] &= \mathbf{F}(\mathbf{x[n]}) \\ \mathbf{w}[n+1] &= \mathbf{A}\mathbf{w}[n] + \mathbf{b}G(\mathbf{x}[n]) \\ z[n] &= \mathbf{c}^T \mathbf{w}[n]. \end{aligned}$$

The matrices $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$ are chosen to represent a minimal realization of the LTI system and $\mathbf{w}[n]$ the state of the filter at time $n$. We also assume that the overall composite system is minimal in the sense that there is no pole zero cancelations between any linear component of the original nonlinear system and the cascaded linear system. The invariance of the Lyapunov exponents under smooth invertible coordinate changes allows us examine certain properties of the filtered signal in this augmented state space with the assurance that the results carry over to the embedded state space.

The collection of Lyapunov exponents of the augmented system can be divided into a set equal to those of the original nonlinear system and a set equal to the log magnitude of the eigenvalues of the matrix $\mathbf{A}$. This follows directly from the relation (2) by computing the gradient of the augmented system. Of course, the eigenvalues of $\mathbf{A}$ correspond to the poles of the system in which we are interested. Further, since the filter is stable, the LTI system contributes only negative exponents.

Convolutional distortion also affects the capacity dimension. This discovery was apparently first reported by Badii *et al* [2] who found that filtering noisy chaotic data to reduce noise caused errors in fractal dimension estimates. Since Badii's original work, several researchers [8, 3] have reported the same phenomenon and proposed hueristic deconvolution procedures. The effect of convolution on the capacity dimension can be examined using the time delay construction described in section 2. The time delay construction defines a transformation of $\mathbb{R}^N$, the state space of the original nonlinear system, to $\mathbb{R}^{\tilde{N}}$ , the space consisting of the reconstructed vectors. We show below that the effect of filtering on the capacity dimension of the observed signal $z[n]$ depends upon the nature of this transformation.

The vectors in the reconstructed space are of the form

$$\begin{aligned} \mathbf{Z}[n] &= \begin{bmatrix} z[n] \\ \vdots \\ z[n-\tilde{N}+1] \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=0}^{\infty} h[i] G(\mathbf{F}^{-i}(\mathbf{x})) \\ \vdots \\ \sum_{i=0}^{\infty} h[i] G(\mathbf{F}^{-\tilde{N}-i}(\mathbf{x})) \end{bmatrix} \\ &\equiv \mathbf{M}(\mathbf{x}). \end{aligned}$$

Again, because of Takens's theorem and our assumptions on $\mathbf{F}$ and $G$, $\mathbf{M}(x)$ will in general be a $LD$ from the attractor in the augmented state space to the reconstructed space for sufficiently large $\tilde{N}$.

For certain impulse responses $h[n]$, $\mathbf{M}(x)$ will actually be an $LD$ from the state space of the original dynamical system to that of the reconstructed vectors. In this case, the invariance of the capacity to LD coordinate changes implies that the capacity dimension of the reconstructed attractor must equal that of the original nonlinear state space system (1). Thus, the class LTI systems which do not affect capacity includes the class of impulse responses for which $\mathbf{M}$ is a $LD$. We develop below a sufficient condition for a LTI system to belong to this class.

The definition of Lipschitz implies that the norm of gradient of a Lipschitz function $\mathbf{M}(x)$ must be bounded for any induced norm. for all vectors $\mathbf{x}$ in the domain of $\mathbf{M}$. The gradient of the embedding transformation is

$$\nabla \mathbf{M}(x) = \left[ \begin{array}{ccc} m_0(x) & \cdots & m_{\tilde{N}-1}(x) \end{array} \right]. \qquad (4)$$

where (with $x_{-i} = \mathbf{F}^{-i}(x)$) the columns are given by

$$m_j(x) = \sum_{i=0}^{\infty} h_i \nabla[\mathbf{F}^{-i-j}(x)] \nabla G(x_{-i-j})$$

We examine conditions for the norm of $\nabla \mathbf{M}$ to be bounded using the maximum absolute column sum norm. Our approach is to bound the columns of the gradient matrix since bounded columns imply a bounded matrix norm.

For any $j$, the norm $m_j(x)$ can be bounded as follows :

$$\|m_j(x)\| \leq \sum_{i=0}^{\infty} |h_i| \|\nabla[\mathbf{F}^{-i-j}(x)] \nabla G(x_{-i-j})\|. \quad (5)$$

Because $G$ is Lipschitz, the norm its gradient $\nabla G$ is bounded, say by $K_G$. Further, from (2) it follows that, if the vector $\mathbf{u}_N$ corresponds to the smallest Lyapunov exponent, $\lambda_N$, the product $\|\nabla[\mathbf{F}^{-i}(x)]\mathbf{u}_N\|$ grows asymptotically like $e^{-i\lambda_N}$. Thus, the sum of right side equation (5) will converge if the sum

$$\sum_{i=0}^{\infty} |h_i| e^{-i\lambda_N}$$

converges. The sum thus converges when $|h_i|$ decays faster than $e^{i\lambda_{min}}$. When $|h_i|$ satisfies this decay condition, each column of $\nabla M(x)$ is bounded and hence the map $M(x)$ is Lipschitz. Note in particular that the sum converges for all FIR filters. It follows that FIR distortion will not increase the capacity.

Based on our assumptions, this decay condition is equivalent to requiring that the log magnitude of the largest eigenvalue of $\mathbf{A}$ – or equivalently, the log magnitude of the largest pole radius of the LTI system– is less than $\lambda_{min}$. Thus, the class of impulse responses with strong decay relative to the chaotic system will not affect capacity dimension. It is interesting to note that one dimension chaotic maps have one positive Lyapunov exponent and no negative ones. This implies that no no rational IIR LTI system will fall in the strong decay class relative to a one dimensional map. In fact, it can be shown using the above result and a result of Young [5], that IIR convolutional distortion will always increase the capacity of the attractors of one dimensional maps.

## 4. DECONVOLUTION

In situations where convolutional distortion is present, our goals include characterizing the distortion and extracting an estimate of the desired signal or its parameters from the distorted observation. Algorithms for signal estimation would be of interest, for example, in applications involving classification of chaotic signals or communication with chaotic signal sets. In other situations, for instance channel characterization, the estimate of the channel may be of interest in itself. This section describes one approach to this deconvolution problem.

One approach to the deconvolution problem is to estimate the parameters of a linear system which approximately inverts the distortion observed by the receiver. Given that the impulse response $b[n]$ of such a system can be estimated , we form the the estimate $\hat{y}[n]$ of the desired signal by filtering the received data $z[n]$. In this scenario, there exists an inherent ambiguity in the signal estimate since delays and scaling factors can just as easily be part of the distortion as the signal. Consequently, we expect to be able to estimate the signal at most to within a constant scale factor $a$ and a constant delay $n_0$.

In order to achieve a high quality estimate, we would like the combined system $(b[n] * h[n])$ to be close in some

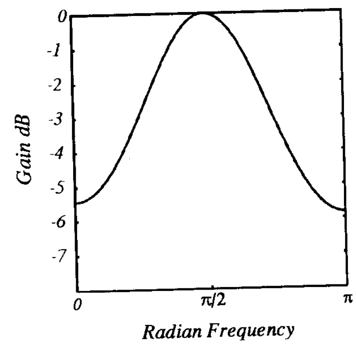



Figure 1: Distorting System Frequency Response

sense to a scaled, shifted unit sample $a\delta[n - n_0]$. If the dynamics of the signal are known, this goal can be achieved by solving a set of nonlinear equations. Specifically, for a $p$ tap equalizer, we must solve the equations

$$G(F^{n-n_0}(x_0)) = \sum_{k=0}^{p-1} b[k]z[n-k].$$

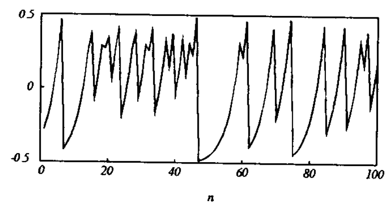
This set of equations can in principle be solved in a least squares sense for the parameters $n_0$, $x_0$, $a$, and the sequence $b[k]$. However, the solution is difficult to obtain directly in part because the positive Lyapunov exponents of the system imply that the the solution will be extremely sensitive in the $x_0$ parameter. In this section we examine another approach to the deconvolution problem assuming much less *a priori* knowledge of the signal generation mechanism.

In many situations the dynamical system responsible for generating the desired signal is unknown. This scenario may occur, for example, when observing physical phenomena such as turbulence with an unknown source. In this case, the deconvolution must be performed using only general *a priori* knowledge. This is referred to as the blind deconvolution problem. Typically, blind deconvolution has been studied in the context of desired signals which are IID stochastic processes. In the stochastic context, the statistical independence of desired signal plays a major role in in the development of algorithms. For example, assuming that the desired signal is white Gaussian noise, a minimum phase equalizer may be designed using linear prediction. Stochastic techniques tend to perform poorly with chaotic signals at least in part because the IID assumption is violated.

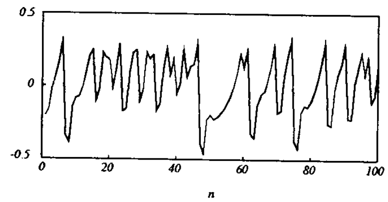
To illustrate the approach to deconvolving a chaotic signal, we consider distorted signals generated by convolving iterates of a one dimensional map with some distorting kernel corresponding to a stable all pole system. In this case, the desired signal is generated deterministically by a scalar difference equation of the form $x[n] = f(x[n-1])$. We observe a distorted version

$$z[n] = \sum_{i=1}^{p} a[i]z[n-i] + x[n]. \quad (6)$$

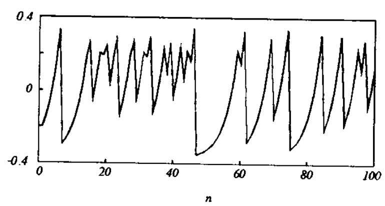
The results of the section 3 show that the capacity dimension of the received signal must be greater than that of the desired signal. This suggests performing the deconvolution by choosing the parameters $b[n]$ to minimize the dimension




(a) Original Waveform $x[n]$




(b) Distorted Waveform $z[n]$




(c) Recovered Waveform $\hat{x}[n]$

Figure 2: Deconvolution Example

of signal estimate $\hat{x}[n]$. An approach similar to this has been proposed by Scargle [9]. We take a somewhat different approach which takes advantage of the deterministic structure of the desired signal. This structure is embodied in short term predictability based upon a single sample of the signal. The idea here is that all-pole convolutional distortion decreases the single sample predictability of the distorted signal relative to the desired signal. This decrease in predictability occurs because a sample of the received signal cannot be expressed as a deterministic function of the previous sample. This line of reasoning leads us to choose the equalizer parameters to minimize the prediction error of the signal estimate.

We consider the error in predicting each value of the received signal using a predictor designed from the data itself. Many techniques for predicting chaotic signals have been described in the literature, see, for example, [11] and the references therein. The following locally constant predictor serves to illustrate the proposed method.

*Given a sequence $z[n]$ $n = 0 \cdots N-1$, we desire to make a one step prediction of a sample, say $z[k]$. Let $j^*$ be the index of the sample nearest in value to $z[k]$ in $z[n]$ $n = 1, \cdots, k-1, k+1, \cdots, N-1$. Then the predicted value is $\hat{z}[k+1] = z[j^*+1]$.*

At each time $n$ the prediction error $e[n] = \hat{\hat{x}}[n+1] - \hat{x}[n+1]$ is computed. The prediction error of the signal is defined as

$$\mathcal{E} = \sum e^2[n]$$

The prediction error of the signal estimate $\hat{x}[n]$ is of course a function of the equalizer parameters. This function is minimized with respect to the filter parameters using standard nonlinear optimization techniques.

The performance of this technique is illustrated in figure 2 with a signal generated by the tent map,

$$x[n+1] = \begin{cases} 4x[n]/3 & 0 \le x[n] \le .75 \\ 4(1-x[n]) & .75 < x[n] \le 1 \end{cases}.$$

The signal $x[n]$ has is fairly lowpass with a power spectrum equivalent to that of white noise through a system with a single pole at $z = .5$ [7]. The frequency response of the distorting filter is shown in figure 1. Figures 2 (a), (b), and (c) show segments of the original, distorted and recovered signals respectively. It is apparent that, with the exception of a scale factor, the distortion apparent in figure 2 (b) has been largely removed.

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