Communication over Fractal Channels

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Abstract

The problem of data transmission over additive Gaussian fractal noise channels is considered. Exploiting an efficient, wavelet-based representation for fractal processes, the problem of coherent detection in Gaussian fractal noise is addressed, from which the optimum receiver for bit-by-bit signaling is obtained. This leads to a multirate modulation strategy that is inherently well-suited for use with the fractal noise channel. Computationally efficient implementations of the transmitter and receiver structures for this system are also developed.

Introduction

Frequently, communication theory focuses on the use of channel models in which the additive noise is white. This is both because white noise is a reasonable model for a variety of broadband noises encountered in many physical systems and because white noise is particularly amenable to analysis.

However, there are a wide range of environments in which the predominant noise is not white but more generally fractal, i.e., the noise is characterized by various degrees of scale-invariance [8] [9]. For instance, such noise is often associated with laser systems, electronic devices and turbulent flow. Consequently, the problem of transmitting data over fractal noise channels is an important one in a variety of optical, electrical, and underwater (acoustic) communication contexts.

This work exploits an efficient representation for fractal noise in terms of orthonormal wavelets bases that was developed in [12]. From this representation, we are able to address problems of optimal receiver design for bit-by-bit signaling in fractal backgrounds. In turn, this suggests an efficient and practical multirate modulation strategy for use with fractal noise channels. Related work on detection in fractal noise is described in [1], while work on the estimation of fractal signals is presented in [13].

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Fractal Noise

The $1/f$ family of fractal noise processes are generally defined [8] as processes whose empirical power spectra are of the form

$$S(f) \sim \frac{\sigma_x^2}{|f|^\gamma} \tag{1}$$

over several decades of frequency f, where $0 < \gamma < 2$ is a parameter typically near unity. Although such power spectra are often not integrable, a variety of interpretations of such spectra have emerged in the literature [4] [8] [9] [12].

A truly enormous range of natural phenomena are well-modeled as $1/f$ processes [8] [9], although the ubiquity itself is not well-understood. Moreover, these processes have a number of important properties, among which is a statistical invariance to scale and persistent long-term correlation structure, that contrast sharply with those of the well-studied class of ARMA processes.

Processes that exhibit only a constant percentage deviation from the nominal $1/f$ characteristic, i.e., processes whose power spectra are bounded according to

$$\frac{k_1}{|f|^\gamma} \leq S(f) \leq \frac{k_2}{|f|^\gamma}, \tag{2}$$

where $0 < k_1 \leq k_2 < \infty$ are arbitrary, also possess the fundamental characteristics of $1/f$ processes. Consequently, the definition of $1/f$ processes can generally be extended to include these *nearly*-$1/f$ processes [13]. Examples of such processes include the *dyadic fractal processes*—processes that are statistically invariant only to changes of scale by factors of two [13].

The fractal noise channels considered in this work are additive noise channels with no intersymbol interference where the noise is a $1/f$ process. While the ideal (infinite bandwidth) fractal noise channel has infinite capacity (since $S(f) \to 0$ as $f \to \infty$), any realistic communication scenario involves a bandwidth constraint. This can be accommodated in our model either by adding a white noise component or by treating the channel as bandlimited. We will generally adopt the latter approach, although it is worthwhile to remark that the former can be readily accommodated using techniques developed in [13].

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Orthonormal Wavelet Bases

Briefly, an orthonormal wavelet transformation of a signal $x(t)$ is defined through the synthesis/analysis equations

$$x(t) = \sum_m \sum_n x_n^m \psi_n^m(t) \longleftrightarrow x_n^m = \int_{-\infty}^{\infty} x(t)\, \psi_n^m(t)\, dt \qquad (3)$$

and has the special property that all basis functions are dilations and translations of a single function: $\psi_n^m(t) = 2^{m/2}\psi(2^m t - n)$, where m and n are the dilation and translation indices, respectively [3]. Since the basic wavelet, $\psi(t)$, has an essentially band-pass Fourier transform, the wavelet transformation is conveniently interpreted in terms of a generalized constant-Q or octave-band filter bank.

A resolution-limited approximation of a signal $x(t)$ in which details on scales smaller than 2^M are discarded can be expressed via the orthonormal expansions

$$A_M x(t) = \sum_{m<M} \sum_n x_n^m \psi_n^m(t) = \sum_n a_n^M \phi_n^M(t), \qquad (4)$$

with the coefficients a_n^M obtained by

$$a_n^M = \int_{-\infty}^{\infty} x(t)\, \phi_n^M(t)\, dt = \left\{ x(t) * \phi_0^M(-t) \right\}\Big|_{t=2^{-M}n}. \qquad (5)$$

The basis functions $\phi_n^m(t)$ also have the property that they are dilations and translations of a single *scaling* function: $\phi_n^m(t) = 2^{m/2}\phi(2^m t - n)$, where $\phi(t)$ has an essentially low-pass Fourier transform.

There is considerable flexibility in the choice of $\phi(t)$, which characterizes the wavelet basis. It is frequently desirable to choose this scaling function to be localized in both time and frequency, and of finite length. There exist a variety of such wavelet bases; see, e.g., Daubechies [3].

The discrete wavelet transform (DWT) defined in terms of the quadrature filter pair

$$h[n] = \int_{-\infty}^{\infty} \phi_0^{-1}(t)\, \phi_n^0(t)\, dt \qquad (6a)$$

$$g[n] = \int_{-\infty}^{\infty} \psi_0^{-1}(t)\, \phi_n^0(t)\, dt \qquad (6b)$$

relates the approximation coefficients a_n^m and detail coefficients x_n^m through the filter-downsample and upsample-filter relations

$$a_n^m = \sum_k h[k-2n] a_k^{m+1} \qquad (7a)$$

$$x_n^m = \sum_k g[k-2n] a_k^{m+1} \qquad (7b)$$

$$a_n^{m+1} = \sum_k \{ h[n-2k] a_k^m + g[n-2k] x_k^m \}. \qquad (7c)$$

See [11] for a discussion of the computational complexity of the DWT.

Wavelet Representations of Fractal Noise

The wavelet-based Karhunen-Loève-like expansion for $1/f$ processes described in [12] is a ***synthesis*** result—that one can construct a class of dyadic fractal processes using wavelet expansions in terms of uncorrelated transform coefficients having the variance progression

$$\mathrm{Var}\, x_n^m = \sigma_m^2 = \sigma^2 2^{-\gamma m} \qquad (8)$$

where γ is the exponent of the nearly-$1/f$ spectrum, and σ^2 is a positive constant proportional to σ_x^2.

At least empirically, there exists a corresponding ***analysis*** result—for a reasonably arbitrary choice of wavelet, there is strong empirical evidence that the wavelet coefficients from all $1/f$ and nearly-$1/f$ processes both obey the variance progression (8) and are weakly correlated both along and across scales. Recently, work such as that of Tewfik, *et al* [10] suggests that this analysis result can be made rigorous.

We shall use these properties of the wavelet expansion to convert the problem of continuous-time communication in fractal noise into a problem of discrete-time communication over parallel white noise channels.

Bit-by-bit Signaling

Let us consider the problem of designing an optimal receiver for bit-by-bit signaling in fractal noise. In particular, consider a collection of mutually orthogonal waveforms $\{s(t-kT)\}$, each of energy E such that the kth bit is represented by $-s(t-kT)$ if the bit is a 0, or by $s(t-kT)$ if the bit is a 1. This corresponds to transmission at rate $R = 1/T$ bits/sec with average power $P = ER$.

Designing the optimal receiver for determining each bit amounts to solving a detection problem involving two equally-likely hypotheses for the received signal $r(t)$:

$$H_0 : r(t) = -s(t) + x(t) \qquad (9a)$$

$$H_1 : r(t) = s(t) + x(t) \qquad (9b)$$

where $x(t)$ represents the fractal noise. Here we assume that the channel is bandlimited, that $s(t)$ is sufficiently low-pass so as to be transmitted undistorted, and that $x(t)$ represents, specifically, the bandlimited fractal noise. We further assume for simplicity that the bandlimit can be expressed as $W = 2^{(\overline{M}-1)}$ Hz for some integer $\overline{M}$.

Using a wavelet-based representation for $x(t)$ for which the corresponding $\phi(t)$ is a reasonably good low-pass filter, we can rewrite (9) as a multichannel hypothesis test:

$$H_0 : r_n^m = -s_n^m + x_n^m \qquad (10a)$$

$$H_1 : r_n^m = s_n^m + x_n^m \qquad (10b)$$

where $m < \overline{M}$ is the channel index and n is the discrete-time index. In each channel, the noise is white with variance (8), and the noise from channel to channel is uncorrelated. For Gaussian noise, it is straightforward to show that the optimal receiver computes

$$\ell = \sum_{m<\overline{M}} \sum_n \frac{r_n^m s_n^m}{\sigma_m^2} \qquad (11)$$

and decides that the received bit is a 1 if $\ell > 0$ and 0 otherwise. The resulting bit-error probability for a system with this receiver can be readily expressed as

$$P(\epsilon) = Q\left(\sqrt{\sum_{m<\overline{M}} \sum_n \frac{(s_n^m)^2}{\sigma_m^2}} \right) \qquad (12)$$

where $Q(\cdot)$ is defined by

$$Q(\alpha) = \frac{1}{2\pi} \int_{\alpha}^{\infty} e^{-\nu^2/2} \, d\nu.$$

Because σ_m^2 is a monotonically decreasing function of m, $s(t)$ can be chosen to minimize (12) subject to the energy constraint E by putting all the energy into the least noisy channel: $m = \overline{M} - 1$. The best achievable bit-error probability using bit-by-bit signaling is then given by

$$P(\varepsilon) = Q\left(\sqrt{\frac{1}{\sigma_m^2} \sum_n (s_n^{\overline{M}-1})^2}\right) = Q\left(\sqrt{\frac{E}{\sigma^2 2^{-\gamma(\overline{M}-1)}}}\right). \quad (13)$$

The results of this section suggest that bit-by-bit signaling can be rather inefficient in many respects, particularly in terms of bandwidth. Indeed, the optimum signaling waveform, expressed as

$$s(t) = \sum_n s_n^{\overline{M}-1} \psi_n^{\overline{M}-1}(t) \quad (14)$$

is essentially band-pass, so that effectively only the upper half of the available bandwidth W is used.

Note that one optimal choice for the signaling waveform for this problem is $s(t) = \sqrt{E}\psi_0^{\overline{M}-1}(t)$. While the use of wavelet basis functions as modulating waveforms is clearly inefficient in this example, the notion of using these bases more generally for modulation has considerable potential, as we show next.

A Multirate Modulation Scheme

In developing an uncoded modulation strategy optimized for the additive Gaussian fractal noise channel, we start by recognizing that for a fixed integer m, we may send data using L_m-level pulse amplitude modulation (PAM) at $b_m 2^m$ bits/sec where $b_m = \log_2 L_m$ using the signaling waveforms $\{\psi_n^m(t)\}$ such that the nth symbol in the stream modulates $\psi_n^m(t)$. Using this modulation, the data is transmitted essentially in the frequency band $2^{m-1} \leq f \leq 2^m$ Hz, the effective frequency support of $\psi_n^m(t)$. With an average energy per symbol E_m, the bit-error rate of such a scheme with the optimum receiver is

$$P_m(\varepsilon) = 2\left(\frac{L_m - 1}{L_m}\right) Q\left(\sqrt{\frac{3E_m}{(L_m^2 - 1)\sigma_m^2}}\right). \quad (15)$$

However, we may add additional, orthogonal channels in parallel by using other values of m and exploiting the orthogonality of the waveforms $\{\psi_n^m(t)\}$. The result is a sequence of data streams occupying consecutive octave-width frequency bands. For each successive value of m, the bandwidth and data rate of the corresponding channel doubles. Note that while the bandwidth constraint W on the system limits the highest channel, the geometric escalation of noise in the channels corresponding to successively lower frequencies ultimately gives rise to a limit on the lowest available channel, $\underline{M}$.

The problem concerning which channels to use and how to distribute power among those channels is well-posed. Specifically, given an average power constraint

$$\sum_{m<\overline{M}} E_m 2^m \leq E \quad (16)$$

and a desired common bit-error rate $P_\varepsilon = P_m(\varepsilon)$ for each of the channels, it is possible to distribute the power among the channels so as to maximize the overall bit rate

$$R = \sum_{m<\overline{M}} b_m 2^m. \quad (17)$$

This optimization problem arises in various forms in such multichannel communication problems, and the solution involves a water-pouring algorithm [2] [7] very similar to that which solves the problem of computing the capacity of the composite channel [6, pp. 343-354].

This multirate modulation scheme can be implemented very efficiently. Specifically, while a straightforward, entirely continuous-time implementation of the transmitter and receiver is possible, it turns out that there is considerable advantage to implementing much of the system in discrete-time by exploiting the structure of the DWT discussed earlier.

The construction of the multiple data streams required for this scheme proceeds as follows. Since there are $M = \overline{M} - \underline{M}$ channels, corresponding to $\underline{M} \leq m < \overline{M}$, and since the symbol rate doubles for each successive channel, the bit stream to be coded can be partitioned into blocks of length

$$\sum_{\underline{M} \leq m < \overline{M}} 2^{m-\underline{M}} b_m \text{ bits,}$$

of which $2^{m-\underline{M}} b_m$ bits are assigned to the mth channel. Continuing this bit assignment strategy among the channels over successive blocks, we construct the M bit streams. Next, each of the bit streams is converted into a stream of L_m-ary symbols by collecting successive groups of b_m bits. Let $\{d_m[n]\}$ represent the stream of symbols for the mth channel, so each $d_m[n]$ takes on one of the L_m values associated with a minimum-energy PAM constellation of average energy E_m.

In terms of the multiple PAM streams described above, the transmitter constructing

$$c(t) = \sum_{\underline{M} \leq m < \overline{M}} d_m[n] \psi_n^m(t) \quad (18)$$

can be implemented as follows. Using the successive upsample-filter-add operations of the DWT synthesis relations (7), the data streams can be combined into a single sequence $c[n]$ as illustrated in Fig. 1. Then, consistent with (4), this composite sequence can be modulated on the low-pass waveform $\phi_0^{\overline{M}}(t)$ to produce

$$c(t) = \sum_n c[n] \phi_n^{\overline{M}}(t) = \sum_n c[n] \phi_0^{\overline{M}}(t - 2^{-\overline{M}} n), \quad (19)$$

which is transmitted over the channel.

The optimum receiver can be implemented in a complementary fashion. First the composite sequence estimate $\hat{c}[n]$ is obtained from the noisy received signal $\hat{c}(t) = c(t) + x(t)$ by low-pass filtering and sampling:

$$\hat{c}[n] = \int_{-\infty}^{\infty} \hat{c}(t) \, \phi_n^{\overline{M}}(t) \, dt = \left\{ \hat{c}(t) * \phi_0^{\overline{M}}(-t) \right\} \Big|_{t=2^{-\overline{M}} n}. \quad (20)$$

Then, using the successive filter-downsample operations of the DWT analysis relations (7), we can extract estimates of the individual data streams $\hat{d}_m[n]$ for $\underline{M} \leq m < \overline{M}$, as shown in Fig. 2. Finally, each of these noisy L_m-ary PAM streams is decoded via a quantization process according to the appropriate minimum distance decision rule.

Concluding Remarks

While the system above has been designed specifically for the additive Gaussian fractal noise channel, it is generally applicable to a much broader range of scenarios. In particular, with the appropriate power distribution among the channels, this system is inherently well-suited to scenarios in which it is more appropriate to treat channel noise and LTI channel characteristics on a log-frequency scale.

Finally, while we have restricted our attention to uncoded modulation, the incorporation of ideas from coded modulation [5] appears to hold considerable promise. Indeed, this, together with a study of the performance of the schemes described in this work, suggests some interesting and important future directions for this work.

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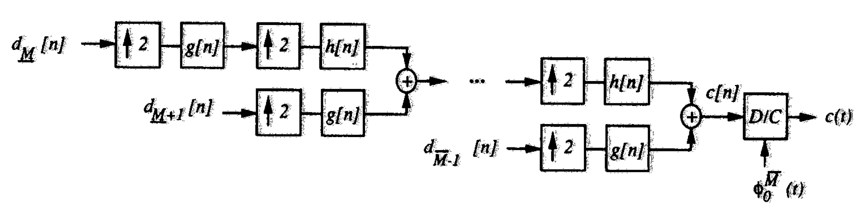

Figure 1: *The transmitter for multirate modulation*

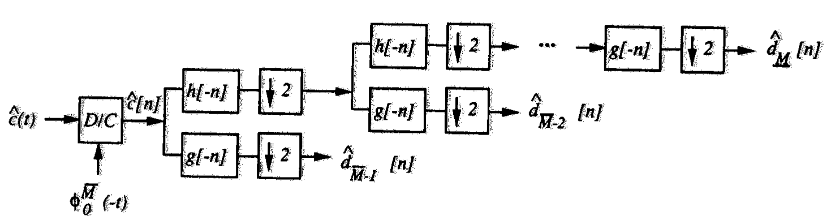

Figure 2: *The optimum receiver for multirate modulation*