Defect Tolerance: Fundamental Limits and Examples

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***Abstract*— This paper addresses the problem of adding redundancy to a collection of physical objects so that the overall system is more robust to failures. In contrast to its information counterpart, which can exploit parity to protect multiple information symbols from a single erasure, physical redundancy can only be realized through duplication and substitution of objects. We propose a bipartite graph model for designing defect-tolerant systems, in which the defective objects are replaced by the judiciously connected redundant objects. The fundamental limits of this model are characterized under various asymptotic settings and both asymptotic and finite-size systems that approach these limits are constructed. Among other results, we show that the simple modular redundancy is in general suboptimal. As we develop, this combinatorial problem of defect tolerant system design has a natural interpretation as one of graph coloring, and the analysis is significantly different from that traditionally used in information redundancy for error-control codes.**

***Index Terms*— Defect-tolerant circuits, bipartite graphs, coloring, combinatorics, worst-case errors.**

I. Introduction

Classical Shannon theory established principles of adding redundancy to data for combating noise and, dually, of removing redundancy from data for more efficient storage. The central object of the classical theory is information, which unlike physical objects, can be freely copied and combined. In fact, the marvel of error-correcting codes is principally based on the counter-intuitive property that multiple unrelated information bits $X_1, \dots, X_k$ can be simultaneously protected by adding "parity-checks" such as

$$Y = X_1 + \cdots + X_k \mod 2. \tag{1}$$

In this example, the added parity-check Y allows the recovery of the original message even if the vector

$$(X_1, X_2, \dots, X_k, Y)$$

undergoes an erasure of an arbitrary element.

Physical objects (e.g., transistors in a chip) may also be subject to erasures (failures) and thus it is natural to ask about ways of insuring the system against failure events. Note, however, that for physical objects operations such as (1) are meaningless. If the failure renders an object completely useless, then protecting against these failures would entail adding spare (redundant) elements. The required operation is to copy and then substitute.[1] It may, therefore, seem that nothing better than simple replication can guard against failures. This paper shows otherwise. Indeed, there exist non-trivial ways to add redundancy as long as the objects' diversity does not exceed their number. That is, if the number of types of objects is smaller than the total number of them.

The objective of this paper is to develop a study of adding redundancy to a physical system where certain objects in the system fail and can only be replaced by substitutes. This paper will explore what are good design choices in this scenario and find fundamental limits for specific settings.

A. Reconfigurable Defect-Tolerant Circuits

To facilitate defining the problem we intend to study, we will first present the application which informed the main model we developed for studying redundancy of physical objects, and that is the application of reconfigurable circuits.

Consider a chip design process, in which the chip is composed of many similar cells (e.g., standard-cell designs of ASICs). Layout of elements in each cell is dictated by the chip manufacturer. Each cell has k input/output buses and k placeholders (nodes) that can be filled in with logic realizing one of q functions. Now because of manufacturing defects, not all k elements in the cell will operate correctly (call these *primary* elements). For this reason, each cell also contains provisions for redundant elements. In particular, there are m placeholders designated as redundant elements. The designer then selects what type of logic to instantiate into these redundant elements. Once the chip is manufactured and placed on the testbed, the testing equipment probes each cell and determines which primary elements are defective. Programmable switches are then used to reconnect input/output buses from the defective primary elements to one of the redundant elements containing the same logic. So the summary of the events happening to each cell during this process is:

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[1] We assume for physical objects, the "error-correction" should provide an exact copy of the object, not merely something functionally equivalent to the object. An example of what is not considered as "correction" would be replacing a cell storing two bits (b_1, b_2) with a cell storing $(b_1, b_1 \oplus b_2)$.

1) Choose the layout of the placeholders and interconnect (these are provisional wires)
2) Choose components (from available collection of possible types) to fill in the primary elements for the reconfigurable circuit
3) Based on primary elements chosen, choose redundant components (from the same collection) to place in redundant placeholders
4) Build the circuit with these components
5) Based on where the defects occurred, reconfigure the interconnect (i.e., enable provisional wires with programmable switches) of the circuit to correct the defects.

In the above summary, step 5 of reconfiguring the defects is a simple operation which requires minimal programming (or switching) of the provisional wiring. This is precisely the advantage of choosing a good layout for the placeholders and provisional wires in step 1. Notice that this layout is universal in the sense that any choice of components in step 2 (which may be arbitrarily dictated by the manufacturer later) should still lead to guarantees on the number of correctable defects. The focus of this work is to study optimal choices of layouts in step 1 so that the rest of the steps in the procedure are possible.

With respect to this application, our goal is to understand what wiring topologies for the layout the chip manufacturer should try to implement in order to attain the optimal trade-off between the number of redundant elements, provisional wires and defect-tolerance. Notice that the two metrics, redundancy and wiring, both correspond to necessary additional resources. Adding redundancy require more silicon area and the provisional wires requires additional metal and programmable switches.[2]

Certainly, there are other procedures and layout constraints we could have chosen to study defects in hardware. For instance, there could be a 2-hop system between the primary elements and redundant elements, decreasing the amount of wiring needed. However, multi-hop interconnects could introduce more latency and make signal propagation delays unpredictable, which is why we do not discuss this in this work, but this is a scenario left for future work (see Section VI-E).

B. Relation to Prior Work

Prior work on the subject of designing digital electronics robust to noise has been traditionally approached with the goal of combating dynamic noise. This is epitomized in the line of work started by von Neumann [1] and contemporary variations [2]. Although significant progress has been made in understanding fundamental limits in von Neumann's model, see e.g., [3]–[11], the practical applications are limited due to a prohibitively high level of redundancy required [12].

Here, instead, we are interested in circuits robust to static manufacturing failures. As illustrated previously, this scenario has the advantage of being able to test which parts of the circuit failed and attempt to configure out (or "wire around") the defective parts. This side information enables significant savings in redundancy [13]. In fact, this method of testing the performance of a device followed by some configuration is rather popular in practice, used in multi-core CPUs [14], analog-to-digital converters [15], sense-amplifiers [16], self-replicating automatons [17], parallel computing [18], [19], etc.

This paper can be seen as an attempt to provide theoretical foundations for the static defect scenario. (In fact, this was our original motivation.)

C. Problem Formulation

We study the following problem formulation: Given k objects ("primary nodes"), connect each one of them to some of the available m spares ("redundant nodes") in such a way that in the event that $t \geq 1$ of the objects fail (originals or spares) the overall system can be made to function after a repair step. Such a repair step consists of replacing each failed primary node with one of the working spares that it is connected to. Each spare can only replace one failed primary node. The key assumptions are 1) the primary nodes are one of q different types (called labels) 2) the spares have to be programmed to one of the q labels *before* the failure events are known and 3) the same connections need to repair all possible choices of labels for the k primary nodes. We are interested in minimizing the number of spare nodes and the number of connections to spare nodes.

Key to our problem formulation is the idea that we want to design the interconnect (wires) before any of the node labels are determined. One might argue that in some applications the interconnect could be allowed to depend on the labeling of primary nodes. Indeed, the latter will be known before the final topology for the chip is made. However, our procedure insists that the interconnect does not depend on this labeling. The advantage of this is that in the reconfigurable circuits framework, the provisional wire-layout is usable regardless of where any element is placed, providing the same defect tolerance guarantee for every possible placement. We seek a *universal* design, which is independent of element types and thus could serve as the new standard cell for all defect-tolerant circuits. We further discuss alternative design methodologies in Section VI.

We intentionally abstracted our problem to a simple model which is more fundamental and relates to other applications needing redundancy for objects and a universal design. For example, instead of parts of a reconfigurable circuit, objects can represent elements in a programmable logic device (e.g., look-up tables (LUTs) in an FPGA). As part of periodical firmware update, a manufacturer assigns values of LUTs (both primary and redundant) without knowledge of locations of device-specific failures. Then, a built-in algorithm for each failed LUT T reconnects it to an adjacent spare LUT R, with the requirement that R and T be equivalent. This built-in local algorithm is a computationally non-demanding way to reconfigure around defective LUTs. Note that the interconnect of the LUTs need to be universal so that any update chosen by the manufacturer (these updates change the configuration of the primary LUTs) has the same guarantee against defects.

[2] There are certainly other metrics (such as geometric constraints or resources to adjust the wiring between primary elements) which are relevant for circuit applications, but we leave consideration of them to future work.

For $q = 2$ our problem is equivalent to finding sparsity vs. edge-size trade-off for (t, t)-colorable hypergraphs, cf. [20]. See Section VI-C. Other applications potentially arise in warehouse planning, operations research, public safety etc. Such applications can be conceived after realizing that our interconnect may be thought of as a transportation network between a collection of "sinks" and "sources" so that each sink can be serviced by at least t sources, where each sink has a type and can be serviced only by sources of the same type.

Expressed mathematically, we are looking for a $k \times m$ bipartite graph with the property that for any q-coloring of the left-side nodes there is a q-coloring of the right-side nodes such that each of the k left-side nodes is connected to at least t nodes of its color. The goal is to find bipartite designs which have efficient trade-off in redundancy m/kt vs. number of edges.

The high-level summary of our main findings is that when $q \geq k$, no strategy is better than straightforward t-fold replication. When $q < k$, there exist designs that provide savings compared to repetition. We fully or partially characterize the fundamental trade-off between redundancy m/kt and the average number of edges per primary node in the following cases:

1) q, t fixed and $k, m \to \infty$;
2) q fixed and $k, m, t \to \infty$;
3) q, k fixed and $m, t \to \infty$.

Perhaps surprisingly, in this (combinatorial) problem it is possible to obtain exact analysis for asymptotics. The organization of the paper is as follows. Section II introduces the problem formally and overviews main results. Section III demonstrates small-size examples that show non-triviality of the problem. Sections IV and V address the trade-off in the regime of fixed t and $t \to \infty$ respectively. Finally, Section VI discusses implications and extensions of our results.

The notation $[n]$ denotes positive integers $1, 2, \ldots, n$. The notation $\mathbb{1}\{\cdot\}$ denotes the indicator function. An underlined letter (e.g, $\underline{x}$) stands for a vector quantity.

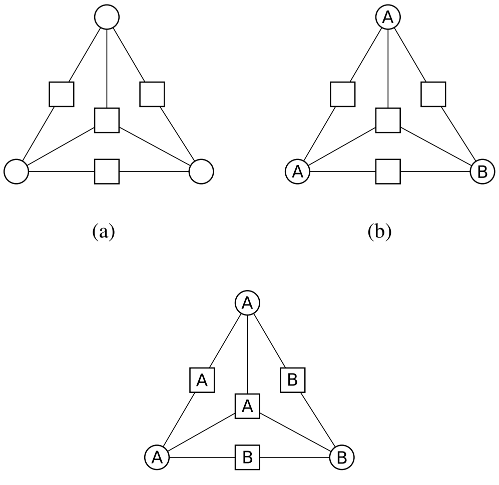
II. Problem Setup and Main Results

A. Defect-Tolerance Model

This paper focuses on bipartite graph designs.[3] The left-side nodes of the bipartite graph are called the primary nodes. These are denoted by circles and there are k of these in the bipartite graph. The right-side nodes are the redundant nodes. These are denoted by squares and there are m of these in the bipartite graph.

Let $\mathcal{X}$ be a finite alphabet where $q = |\mathcal{X}|$.

Definition 1: Fix an alphabet of labels $\mathcal{X}$ with size q. A $k \times m$ bipartite graph is called a t-defect correcting design *if for any labeling of k primary nodes with elements of $\mathcal{X}$ there exists a labeling of m redundant nodes with elements of $\mathcal{X}$ such that every primary node labeled $x \in \mathcal{X}$ has at least t neighbors labeled x. We will call such a graph a $(k, m, t, E)_q$-design, with E denoting the number of edges. (See Fig. 1 for an illustration.)*



(a) (b)

(c)

Fig. 1. Example of a 2-defect correcting design for an alphabet $\mathcal{X} = \{A, B\}$ of size $q = 2$. The design is bipartite with the circles representing the left-side nodes and the squares representing the right-side nodes. Fig. 1(b) shows a labeling of the primary (circle) nodes. To each such labeling, we strategically choose a labeling of the redundant (square) nodes, so that each primary node has $t = 2$ neighbors with matching labels (see Fig. 1(c)). Since such a choice is possible for each of the $2^3 = 8$ possible labelings of primary nodes, we conclude that this design is $t = 2$ defect correcting in the sense of Definition 1 and is a $(3, 4, 2, 9)_2$-design.

This paper is devoted to characterizing the fundamental trade-off between the two basic parameters of t-defect correcting designs: redundancy and wiring complexity. The redundancy of a $(k, m, t, E)_q$-design is $\rho = m/(kt)$. The wiring complexity (or average degree per defect) of a $(k, m, t, E)_q$-design is $\varepsilon = E/(kt)$. This trade-off can be encoded in a two-dimensional region:

Definition 2: For a fixed q and $t \geq 1$ we define the region $\mathcal{R}_t$ as the closure of the set of all achievable pairs of (ε, ρ):

$$\mathcal{R}_t \triangleq \text{closure}\left\{\left(\frac{E}{kt}, \frac{m}{kt}\right) : \exists (k, m, t, E)_q\text{-design}\right\}. \qquad (2)$$

To interpret between Definition 1 and reconfigurable circuits (and other applications), we present the following association.

Proposition 3: An interconnect for a reconfigurable circuit can tolerate any t manufacturing defects for any choice of primary nodes if and only if the interconnect is a t-defect correcting design.

Proof: If the interconnect corrects fewer than t defects, there is some primary node labeling where any labeling of the redundant nodes would result in some primary node with label x having fewer than t neighbors with the same label x. If this primary node and all its matching neighbors have defects, then the defect in the primary node cannot be corrected.

If the interconnects is a t-defect correcting design, with the correct redundant node labeling scheme, any primary node labeled x has t redundant neighbors with the same label x. If there are only t defects, either this primary node is

[3] The word choice of "design" is not intended to relate to the notion of combinatorial (Steiner) designs or any other established mathematical definitions.

working, or this primary node has a defect and at most $t-1$ of its neighbors have a defect or are used to correct another primary node. In the latter case, there is at least one redundant node with label x available which can be used to replace this primary node. □

As noted earlier, our performance metrics, ρ and ε, correspond to the extra silicon area and wiring (and fan-out) required respectively for defect-tolerance.

Before proceeding further, we summarize some of the basic properties of regions $\mathcal{R}_t$.

Proposition 4 (Properties of $\mathcal{R}_t$): Regions $\mathcal{R}_t$ satisfy the following:

1) *$(\varepsilon, \rho) \in \mathcal{R}_t$ iff there exists a sequence of $(k, m, t, E)_q$-designs with $\frac{E}{kt} \to \varepsilon$, $\frac{m}{kt} \to \rho$ as $k, m \to \infty$;*
2) *If $(\varepsilon, \rho) \in \mathcal{R}_t$ and $\varepsilon' \geq \varepsilon$, $\rho' \geq \rho$ then $(\varepsilon', \rho') \in \mathcal{R}_t$;*
3) *$\mathcal{R}_t$ are closed convex subsets of $\mathbb{R}_+^2$;*
4) *We have*

$$\limsup_{t\to\infty} \mathcal{R}_t = \text{closure}\left\{\bigcup_{t\geq 1} \mathcal{R}_t\right\} \triangleq \mathcal{R}_\infty. \tag{3}$$

5) *The limiting region $\mathcal{R}_\infty$ is also a closed convex subset of $\mathbb{R}_+^2$ characterized as*

$$\mathcal{R}_\infty \triangleq \text{closure}\left\{\left(\frac{E}{kt}, \frac{m}{kt}\right) : \exists (k, m, t, E)_q - design\right\}. \tag{4}$$

See Section IV-B for proofs.

B. Preview of Main Results for Binary Alphabet

Notice the wiring complexity and redundancy metrics represent the linear scaling between the quantities E and m respectively with the product kt. Designs which satisfy Definition 1 must have the number of redundant nodes and number of edges grow linearly with the product kt. The goal of our results is to find a tight understanding of the coefficient in this linear scaling.[4]

In this section, for the purpose of illustration, we give a summary of our results for the case of binary alphabet $\mathcal{X}$ (i.e., $q = 2$). The rest of the paper will present various bounds and constructions which apply to general alphabet sizes, (i.e., arbitrarily values of q).

There are three separate results which are the main contributions of this paper. One is characterizing the region $\mathcal{R}_t$ in the regime where t is small, specifically for values where $t = 1$ and $t = 2$. The second main result is characterizing the region $\mathcal{R}_\infty$, which corresponds to the limit of regions $\mathcal{R}_t$ when t tends to infinity. The third is characterizing the result when the number of primary nodes k is finite (the first two results have infinite k) and t tends to infinity.

The theorem for the small t case is the following:

Theorem 5: For binary alphabet $\mathcal{X}$, if $t = 1$ or $t = 2$, we have

$$\mathcal{R}_t = \{(\varepsilon, \rho) : \varepsilon \geq 1, \rho \geq 0 \text{ and } \varepsilon \geq 2 - \rho\}. \tag{5}$$

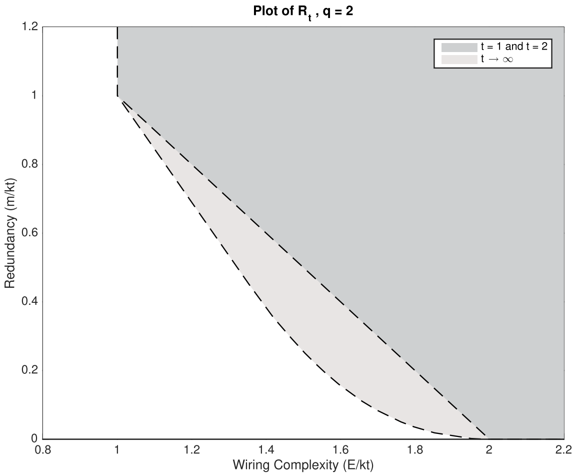


Fig. 2. Achievable regions for redundancy and wiring complexity trade-off when $q = 2$. Regions $\mathcal{R}_1$ and $\mathcal{R}_2$ are shown in darker gray. Region $\mathcal{R}_\infty$ includes lighter and darker gray areas. All other regions $\mathcal{R}_t$ lie between $\mathcal{R}_1$ and $\mathcal{R}_\infty$. The boundary of the region $\mathcal{R}_\infty$ is calculated using the methods in Appendix E.

This will be proved in Section IV-D. The immediate conclusion from this result is that the designs for $t = 1$ and $t = 2$ achieve the same number of redundant nodes and edges needed per primary node per defect asymptotically over k. The region in Theorem 5 has two corner points. We will also discuss the designs which attain these corner points.

The theorem for the asymptotic t case is the following:

Theorem 6: Let $\mathcal{X}$ be a binary alphabet. The region $\mathcal{R}_\infty$ defined in (3) *is the closure of the set of points (ε, ρ) defined as follows. For every distribution P_S on $\mathbb{Z}_+$ with finite support, we define*

$$\varepsilon = \frac{\mathbb{E}[S]}{F(P_S)}, \quad \rho = \frac{1}{F(P_S)}, \tag{6a}$$

where

$$F(P_S) \triangleq \min_{0\leq\lambda\leq 1} \max_{0\leq f(\cdot,\cdot)\leq 1} \min\left\{\mathbb{E}\left[\frac{L_0}{\lambda} f(L_0, L_1)\right],\right. \\ \left.\mathbb{E}\left[\frac{L_1}{1-\lambda}(1 - f(L_0, L_1))\right]\right\} \tag{6b}$$

where the expectations are over $S \sim P_S$ and given S the distribution of $L_1 \sim Bino(S, \lambda)$ and $L_0 = S - L_1$.

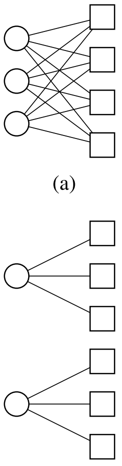
This theorem parametrically characterizes $\mathcal{R}_\infty$ in terms of the function $F(P_S)$, which is evaluated on every P_S with finite support. Note that evaluation of the bound (6a) is non-trivial as we will discuss in Section V-E.

The generalization of Theorem 6 to larger alphabet sizes is given by Theorem 19 and is developed in Section V. Here the designs achieving the best trade-off are more complicated than those associated with Theorem 5. We call them *subset designs* and develop them in Section III-B.

The resulting achievable regions for Theorem 5 and Theorem 6 are depicted in Fig. 2. Via these results we can determine at any fixed redundancy level, how many connections are necessary. For example, at redundancy level 10%, the figure indicates that there exists designs which:

- correct 1 defect if each primary node is connected on average to about 1.9 redundant nodes

[4] For all our results, q is always fixed. How wiring complexity and redundancy scales with q is left for future work.


(a)

(b)

Fig. 3. Two elementary designs. (a) Example of complete design. (Design written as $K(3,4)$). This design is 2-defect correcting for $q=2$ and 4-defect correcting for $q=1$. (b) Example of repetition design. (Design written as $2K(1,3)$). This design corrects 3 defects for any q.

- correct 2 defects if each primary node is connected on average to about 1.9×2 redundant nodes
- correct 10^3 defects if each primary node is connected on average to about 1.7×10^3 redundant nodes.

Immediate from Fig. 2 is that the region $\mathcal{R}_\infty$ contains the regions $\mathcal{R}_1$ and $\mathcal{R}_2$ implying that increasing the number of defects t allows for lower redundancy and wiring complexity (recall both these quantities are divided by t). In this sense, it is more efficient to correct more defects.

According to (4) all regions $\mathcal{R}_t$ will lie between $\mathcal{R}_1$ and $\mathcal{R}_\infty$, approaching the latter as $t \to \infty$. It is perhaps surprising that unlike most known asymptotic combinatorial problems, this one (for $t \to \infty$) admits a relatively simple solution.

The third and the more practically useful result is the characterization of the achievable regions for asymptotic t but with finite k. This is developed in Section V-F.

III. Examples of Good Designs

Before developing the main results, we will first introduce a few basic designs and analyze their performances. Some of these examples play major roles in subsequent developments.

We denote by $K(k,m)$ a complete bipartite graph with k primary nodes (circles) and m redundant nodes (squares). The two most basic designs are the following:

1) *Complete designs:* $K(k,qt)$ (recall that $q = |\mathcal{X}|$) is t-defect correcting. (Just label redundant nodes to hold t copies of each value $\mathcal{X}$. No matter how the primary nodes are labeled, each primary node will be connected to t redundant nodes with the same label as itself.) See Fig. 3(a) for illustration.
2) *Repetition designs:* $K(1,t)$ is capable of correcting t defects over an arbitrary alphabet. (Just label all t redundant nodes the same label as the neighboring primary node.) Taking k disjoint copies of $K(1,t)$, denoted by $kK(1,t)$, we get a repetition design achieving $\rho = \varepsilon = 1$. See Fig. 3(b) for illustration.

If we take $k \to \infty$, the complete design achieves $\varepsilon = q$ and $\rho = \frac{qt}{kt} \to 0$ for any fixed t and q, which is the best possible trade-off given the value of ε. For finite k, however, the complete design is not the design with the minimal number of edges: it is possible to remove some of the edges and still maintain a t defect correcting property, as we will show in the next subsection.

The repetition design uses the minimal number of edges (since any primary node needs at least t edges in order to be a t-defect correcting design). If all primary nodes have exactly t edges, then it is necessary for each primary node to have a distinct neighborhood, illustrating that the repetition design achieves the best trade-off at minimal wiring complexity.

A. Smallest Non-Trivial Designs

We now present designs which have the fewest number of edges given some fixed number of primary nodes k, redundant nodes m, and defect tolerance t.

If $k \le q$ then all primary nodes can have different values and thus one is forced to use the repetition design $kK(1,t)$ to correct t defects. For $k = q+1$ the question becomes more interesting. First, notice that the minimal possible m equals qt (this is achieved by the complete design and cannot be reduced since t nodes with the same label can have defects and this can occur for each of the q different labels). However, some of the edges can be removed from the complete design while still preserving the number of defects corrected.

The optimal designs with $k = q+1$, $m = q$ and $t = 1$ are as follows:

- Binary alphabet ($q = 2$): $k = 3$, $m = 2$ with 5 edges. See Fig. 4(a).
- Ternary alphabet ($q = 3$): $k = 4$, $m = 3$ with 8 edges. See Fig. 4(b). (There exist two non-isomorphic optimal designs. Fig. 4(b) shows the symmetric one.)
- Quaternary alphabet ($q = 4$): $k = 5$, $m = 4$ with 12 edges. See Fig. 4(c). (There exist multiple non-isomorphic optimal designs. Only one is shown.)

The optimal designs with $k = q+1$, $m = 2q$ and $t = 2$ are as follows:

- Binary alphabet ($q = 2$): $k = 3$, $m = 4$ with 9 edges, see Fig. 5(a). This design is what we call the *Hamming block*. Fig. 1 shows how it can correct 2 defects. We will discuss its optimality in Corollary 27.
- Ternary alphabet ($q = 3$): $k = 4, m = 6$ with 15 edges, see Fig. 5(b). (There exist two non-isomorphic optimal designs. Fig. 5(b) shows the symmetric one.)
- Quaternary alphabet ($q = 4$): $k = 5, m = 8$ with 21 edges, see Fig. 5(c).

Some of these designs were found analytically and others by exhaustive search. None of these designs are at the performance boundary of any $\mathcal{R}_t$ regions. To obtain designs that near the optimal trade-off boundary, we need to use a larger number of primary and redundant nodes (see Proposition 15). However, a few of these designs, like the Hamming block in Fig. 5(a), achieve the best trade-off when restricted to the finite k setting (as we will develop in Section V-F).

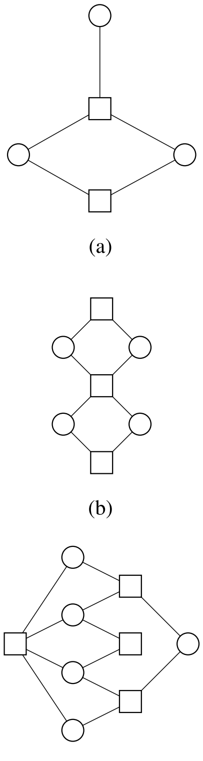


Fig. 4. Smallest non-trivial 1-defect correcting designs. (a) $q = 2$. (b) $q = 3$. (c) $q = 4$.

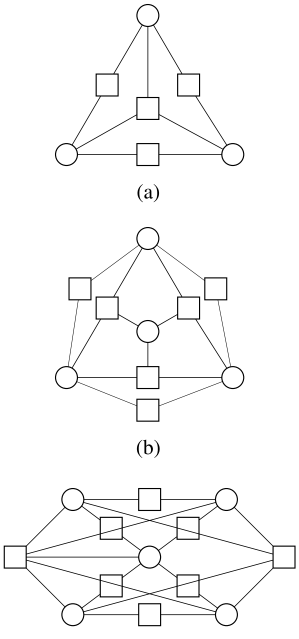


Fig. 5. Smallest non-trivial 2-defect correcting designs. (a) $q = 2$. (b) $q = 3$. (c) $q = 4$.

B. Subset Designs

Designs that form a key ingredient of our asymptotic (i.e., large t) constructions are subset designs. A subset design $S(k, s)$ is a bipartite graph with k primary nodes and $m = \binom{k}{s}$ redundant nodes, each connected to a distinct s-subset of $\{1, \ldots, k\}$. Note that the degree of each primary node is $\binom{k-1}{s-1}$.

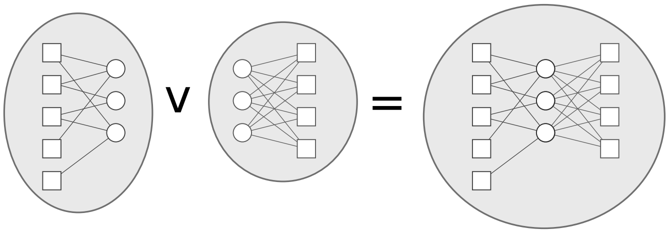


Fig. 6. Example of merging two designs.

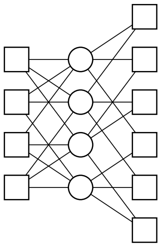


Fig. 7. Example of a subset design. This design is $S(4, 3) \vee S(4, 2)$. The redundant nodes corresponding to $S(4, 3)$ are shown on the left side and those corresponding to $S(4, 2)$ are shown on the right side.

In general, we allow subset designs to have multiple and possibly different subset sizes. For two values s_1 and s_2, where $s_1, s_2 \in [k]$, a bipartite graph $S(k, s_1) \vee S(k, s_2)$ is defined to be the result of identifying the k primary nodes in two disjoint copies of $S(k, s_1)$ and $S(k, s_2)$. The resulting graph has k primary nodes and $m = \binom{k}{s_1} + \binom{k}{s_2}$ redundant nodes. We call the operation ($\vee$) graph merging, which we state more precisely below. We will develop the properties of merging later.

Definition 7 (Merging): For any collection of designs G_j on the same number of primary nodes k, the merging of G_j, denoted $G = \bigvee_j G_j$ is a graph formed by taking disjoint copies of G_j and identifying primary nodes.

See Fig. 6 for an illustration of merging.

Definition 8 (Subset Design): Given k and (not necessarily distinct) positive integers $s_1, s_2, \ldots, s_r \in [k]$,

$$S(k, s_1) \vee S(k, s_2) \vee \cdots \vee S(k, s_r) \tag{7}$$

is a subset design with k primary nodes and $m = \sum_{j=1}^{r} \binom{k}{s_j}$ redundant nodes.

For example, the Hamming block, Fig. 5(a), is $S(3, 2) \vee S(3, 3)$, the repetition design is $S(k, 1) \vee \cdots \vee S(k, 1)$ (t times) and the complete design is $S(k, k) \vee \cdots \vee S(k, k)$ (qt times). Fig. 7 shows the subset design $S(4, 3) \vee S(4, 2)$.

Subset designs are characterized by the following property:

Definition 9 (Permutation Invariance): A design is called permutation invariant *if there exists a group of bipartite-graph automorphisms (thus preserving the left/right partition) that acts as the full symmetric group S_k on primary nodes.*[5]

Proposition 10: A design is permutation invariant if and only if it is a subset design.

[5] For those not familiar with bipartite-graph automorphisms: Consider identifying each primary node and redundant node in the design with a distinct number. Primary node number i is connected to some set of numbered redundant nodes M_i. We can equivalently say a design is permutation invariant if for all possible permutations of the numbers of the primary nodes, there is a way to permute the numbers of the redundant nodes, so that the new design still has primary node i connected to the set of redundant nodes M_i.

Proof: Invariance of subset designs is clear. Conversely, given a permutation invariant design and an integer $s \geq 1$, consider the subgraph induced by all degree-s redundant nodes and their neighborhoods. By permutation invariance this subgraph must contain all k primary nodes and itself be permutation invariant (since automorphisms preserve degrees of nodes). Therefore, every s-subset of the primary nodes must appear as a neighborhood of n redundant nodes for some integer n. This degree-s subgraph corresponds to merging of n copies of $S(k, s)$ and the original graph is a merging of degree-s subgraphs. □

The number of redundant nodes used in subset designs is large and therefore it should be able to correct many defects. We will find sharp estimates for the defect-correcting properties of subset designs later (Proposition 22 below), but for now we can give a simple order-of-magnitude result:

Proposition 11: Fix alphabet $\mathcal{X}$ and size $s \geq 1$. As $k \to \infty$ the design $S(k, s)$ corrects $t = \Theta(k^{s-1})$ defects.

Proof: We know that $t = O(k^{s-1})$, since each primary node has at most $\binom{k-1}{s-1} = O(k^{s-1})$ neighbors. To show that $t = \Omega(k^{s-1})$, fix a labeling of the primary nodes with the elements of $\mathcal{X}$. Consider the following procedure for labeling redundant nodes. First we declare an element of $x \in \mathcal{X}$ to be *rare* if the number of primary nodes labeled x is less than $\frac{k}{q}$. Now each redundant node is labeled the value $x \in \mathcal{X}$ if either all of its neighbors have label x or if x is the only rare label in its neighborhood. (Some redundant nodes may not be labeled, but the contribution from these nodes can be disregarded for this particular order of magnitude result.) To see that this is an labeling that corrects $\Omega(k^{s-1})$ defects, simply notice that a non-rare primary node labeled x has at least $\binom{k/q-1}{s-1} = \Omega(k^{s-1})$ neighboring redundant nodes with all neighbors labeled x. Similarly, for any choice of non-rare label x, each rare-labeled primary node has at least $\binom{k/q-1}{s-1} = \Omega(k^{s-1})$ neighboring redundant nodes connected to it such that all other neighbors of this redundant node is labeled x. Since x is non-rare, these $\Omega(k^{s-1})$ are labeled the value of the rare primary node. □

As we will see, subset designs turn out to be optimal for achieving the boundary of $\mathcal{R}_\infty$. In other words, they can be tuned to get the optimal speed of growth for redundancy and wiring complexity as $t \to \infty$.

IV. Bounds for Finite t

In this section we prove a number of basic results, which will lead to the proof of Theorem 5. We will first show how two basic operations, copying and merging, can be used to combine existing designs into a new design with certain properties. Using these operations, we then proceed to prove the claims in Proposition 4.

Using the convexity results from Proposition 4, we show achievability for Theorem 5. Following the achievability, we show the converse for Theorem 5 which uses a technique we call covering.

A similar result for the achievable region for ternary alphabet is stated at the end of the section.

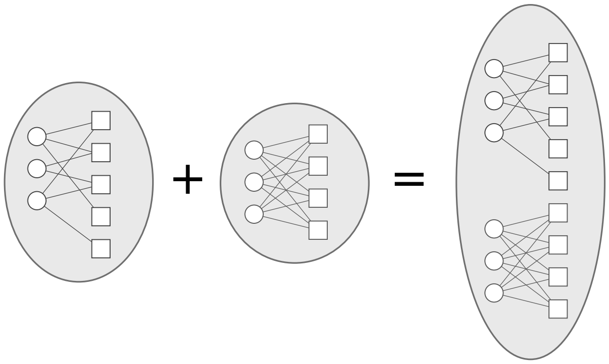


Fig. 8. Example of the copying operation on two designs.

A. Two Basic Operations on Designs

Definition 12 (Copying): A disjoint union, which we call copying, of two designs G_1 and G_2 is denoted by $G_1 + G_2$. A disjoint union of a collection of designs G_j is denoted by $\sum_j G_j$. A disjoint union of n identical designs G is denoted as nG.

Copying is simply the idea of creating a new design from two designs where each design is a disjoint component of the new design. Defining this operation formally is useful for our analysis. See Fig. 8 for an illustration of copying.

Proposition 13 (Copying): Consider $(k_j, m_j, t, E_j)_q$-designs G_j. Then $\sum_j G_j$, forms a $(\sum k_j, \sum m_j, t, \sum E_j)_q$-design.

The proof is clear after realizing that the number of defects corrected does not change while all other parameters must add. We note here that the values of ε and ρ for $G_1 + G_2$ is a convex combination of those of G_j. That is

$$\rho = \frac{k_1}{k_1 + k_2}\rho_1 + \frac{k_2}{k_1 + k_2}\rho_2 \tag{8}$$

$$\varepsilon = \frac{k_1}{k_1 + k_2}\varepsilon_1 + \frac{k_2}{k_1 + k_2}\varepsilon_2 \tag{9}$$

where ρ_j and ε_j refer to $\frac{m_j}{k_j t}$ and $\frac{E_j}{k_j t}$ of G_j respectively.

Proposition 14 (Merging): Consider $(k, m_j, t_j, E_j)_q$-designs G_j and $G = \bigvee_j G_j$ (see Definition 7). Then G is a $(k, \sum_j m_j, \sum_j t_j, \sum_j E_j)_q$-design.

(Note that it is possible that the merged design $G = \bigvee_j G_j$ can correct more than $\sum_j t_j$ defects.) The proof is clear after realizing that the same labeling G_j used for redundant nodes to be t_j correcting for a specific labeling of the k primary nodes can be used in the merged design. As an example, we note that merging a design with itself, i.e., $G \vee G$, doubles all the parameters except k. However, the wiring complexity and redundancy stays constant. This will be the basis for showing convexity of $\mathcal{R}_\infty$, cf. (4).

B. Proof of Proposition 4

With the help of the two basic operations, we can prove the convexity of $\mathcal{R}_t$ and $\mathcal{R}_\infty$, as well as the other properties claimed in Proposition 4.

Proof of Proposition 4: Claim 1: From the definition of closure, $(\varepsilon, \rho) \in \mathcal{R}_t$ if and only if there is a sequence of points $\{(\varepsilon_i, \rho_i)\}_i \in \mathcal{R}_t$ approaching (ε, ρ). Each (ε_i, ρ_i) must be associated with a $(k_i, m_i, t, E_i)_q$-design G_i, where $m_i = \rho_i k_i t$ and $E_i = \varepsilon_i k_i t$. To show that $k, m, E \to \infty$, we can copy G_i with itself n_i times, where n_i is chosen so that $n_i k_i, n_i m_i, n_i E_i \to \infty$.

Claim 2: For any $(\varepsilon, \rho) \in \mathcal{R}_t$, if there is a $(k, \rho kt, t, \varepsilon kt)$-design G, then we can copy G with itself multiple times to get a $(k', \rho k't, t, \varepsilon k't)$-design G' where k' is arbitrarily large. We can always add more redundant nodes or more edges to G' (this is possible since $\rho k't$ can be arbitrarily large and adding a finite number of redundant nodes does not change the redundancy) to G' to get a design with parameters (ε', ρ'). If (ε, ρ) is a limit point achieved by a sequence of designs, we can always similarly add more redundant nodes and edges to each design in the sequence that attains the limit.

Claim 3: This holds using copying from Proposition 13.

If a pair of values (ε_1, ρ_1) and (ε_2, ρ_2) are in $\mathcal{R}_t$, there are sequences $(\varepsilon_{1,i}, \rho_{1,i}) \to (\varepsilon_1, \rho_1)$ and $(\varepsilon_{2,i}, \rho_{2,i}) \to (\varepsilon_2, \rho_2)$, where for each i there exists a $(k_{1,i}, \rho_{1,i} k_{1,i} t, t, \varepsilon_{1,i} k_{1,i} t)_q$-design $G_{1,i}$ and a $(k_{2,i}, \rho_{2,i} k_{2,i} t, t, \varepsilon_{2,i} k_{2,i} t)_q$-design $G_{2,i}$. For any $0 \le \alpha \le 1$, we can find a sequence of rational numbers $\alpha_i = \frac{p_i}{q_i}$ where $p_i, q_i \in \mathbb{Z}_+$ and $\alpha_i \to \alpha$. The copy $k_{2,i} p_i G_{1,i} + k_{1,i}(q_i - p_i) G_{2,i}$ achieves the point $(\varepsilon_i, \rho_i) = (\alpha_i \varepsilon_{1,i} + (1-\alpha_i)\varepsilon_{2,i}, \alpha_i \rho_{1,i} + (1-\alpha_i)\rho_{2,i})$ in $\mathcal{R}_t$ and $(\varepsilon_i, \rho_i) \to (\alpha\varepsilon_1 + (1-\alpha)\varepsilon_2, \alpha\rho_1 + (1-\alpha)\rho_2)$.

Claim 4: Any point (ε, ρ) in $\mathcal{R}_\infty$ and any point in closure, $\left\{\bigcup_{t=1}^{\infty} \mathcal{R}_t,\right\}$ must both be the limit of some sequence of $(k_i, m_i, t_i, E_i)_q$-designs. To see that $\mathcal{R}_\infty = \limsup \mathcal{R}_t$, by merging in Proposition 14, for any t, we have $\mathcal{R}_t \subset \mathcal{R}_{2t} \subset \mathcal{R}_{4t} \subset \mathcal{R}_{8t} \ldots$.

Claim 5: This holds using merging from Proposition 14. Given two designs G_1 and G_2, where G_1 is a $(k_1, \rho_1 k_1\, t_1, t_1, \varepsilon_1 k_1\, t_1)_q$-design and G_2 is a $(k_2, \rho_2 k_2\, t_2, t_2, \varepsilon_2 k_2\, t_2)_q$-design, if we want to create a design G with the parameter $(\alpha\varepsilon_1 + (1-\alpha)\varepsilon_2, \alpha\rho_1 + (1-\alpha)\rho_2)$ for $\alpha = \frac{p}{q}$ where $p, q \in \mathbb{Z}_+$, then we can let

$$G = pk_2\left(\bigvee_{i=1}^{t_2} G_1\right) + (q-p)k_1\left(\bigvee_{i=1}^{t_1} G_2\right). \tag{10}$$

From here on, the proof proceeds similarly to the proof of Claim 3. □

C. Elementary Achievability

From the previous propositions, we can immediately make statements on what each region $\mathcal{R}_t$ must contain. Recall that for any t, the point $(1, 1)$ in $\mathcal{R}_t$ is achievable using the repetition design. The point $(q, 0)$ is asymptotically achievable using the complete design. Thus, the line of points between $(1, 1)$ and $(q, 0)$ is achievable by interpolating between the repetition design $K(1, t)$ and the complete design $K(k, qt)$. We summarize this below:

Proposition 15: The following region is achievable for any $t \ge 1$ and $q \ge 2$:

$$\mathcal{R}_t^{(K)} \triangleq \{(\varepsilon, \rho) : \varepsilon \ge q + (1-q)\rho, \varepsilon \ge 1, \rho \ge 0\}. \tag{11}$$

Furthermore, every point such that $(\varepsilon - 1)$ is a multiple of $(q - 1)$ can be achieved via a design with constant degree ε primary nodes.

Proof: The corner points $(1, 1)$ and $(q, 0)$ are achieved by the repetition design and the complete design, respectively. By Proposition 4 the region $\mathcal{R}_t$ is convex and hence must contain $\mathcal{R}_t^{(K)}$. All rational points near the boundary of $\mathcal{R}_t$ are achieved by $r_1\ K(1, t) + r_2\ K(k, qt)$ for some choice of integers r_1, r_2 and k.

In order to get a design where the primary nodes have regular degree, we can combine the repetition design and complete design by merging. Find two integers t_1, t_2 where $t_1 + t_2 = t$. The combination $kK(1, t_1) \vee K(k, qt_2)$ also achieves the boundary point at $\varepsilon = (t_1 + qt_2)/t$ as $k \to \infty$. This proves the last sentence of the Proposition 15. □

The region $\mathcal{R}_t^{(K)}$ is an inner bound on all achievable regions, but for $q = 2$ and $t = 1, 2$ the region $\mathcal{R}_t^{(K)}$ happens to be tight and is the region plotted in Figure 2.[6]

D. Covering Converse

This section presents a general converse bound which holds for all $\mathcal{R}_t$ and all q, but in particular this converse shows that $\mathcal{R}_2^{(K)}$ is tight for $q = 2$.

Theorem 16: Fix $q = |\mathcal{X}|$, t and suppose $(\varepsilon, \rho) \in \mathcal{R}_t$. Then there exists $\pi_t, \pi_{t+1}, \ldots, \pi_{qt} \ge 0$ satisfying

$$\frac{1}{t}\sum_{j=t}^{qt} j\pi_j \le \varepsilon \tag{12}$$

$$\sum_{j=t}^{qt} \pi_j = 1 \tag{13}$$

$$\sum_{j=t+1}^{qt} \pi_j \log_q \lfloor j/t \rfloor \ge 1 + (t-1)\pi_t - \rho t. \tag{14}$$

In other words the smallest achievable ε for a given ρ is lower bounded as

$$\varepsilon^*(\rho, t) \ge \min\left\{\frac{1}{t}\sum_{j=t}^{qt} j\pi_j : \pi_j \ge 0 \textit{ satisfy } (13)\text{-}(14)\right\} \tag{15}$$

Proof: The key idea of this proof is to look at how the degree of primary nodes relates to whether a design can correct defects for all sequences of labelings. Let us define π_j, $j = t, t+1, \ldots, qt - 1$ to be the fraction of primary nodes with degree j. (Notice that every primary node clearly should have degree at least t.) Define π_{qt} to be the fraction of primary nodes of degree qt or larger. The fact that this satisfies (12)-(13) is obvious. We only need to show (14).

To that end, for each labeling $r^m \in \mathcal{X}^m$ of redundant nodes let $\mathcal{G}_t(r^m)$ be the set of primary node labelings for which conditions of Definition 1 are satisfied (we say that r^m covers

[6] Note that in the worst case, the rate of convergence to get ϵ close to a point on the boundary of $\mathcal{R}_t^{(K)}$ requires k to be on the order of $\frac{1}{\epsilon}$. This occurs when trying to achieve the boundary point $(qt, 0)$. On the other hand, achieving point the boundary point $(1, 1)$ can be done with $k = 1$. For other points on the boundary away from $(qt, 0)$, it is not clear what the best rate of convergence is.

the labelings in $\mathcal{G}_t(r^m)$). The design is t-defect correcting if and only if every possible labeling is covered by some r^m. We can count the number of primary node labelings covered by some r^m and make sure this is equivalent to all possible primary node labelings. Thus, a design is t-defect correcting if and only if

$$\left| \bigcup_{r^m \in \mathcal{X}^m} \mathcal{G}_t(r^m) \right| = |\mathcal{X}|^k = q^k. \tag{16}$$

We are aiming to apply the union bound to the right-hand side to get inequality (14). Before doing so we make the following observation.

Two primary nodes of degree t should have disjoint neighborhoods (otherwise labeling them different values clearly violates Definition 1). Thus $\mathcal{G}_t(r^m)$ is empty unless each such neighborhood has a constant label. This shows that for the $tk\pi_t$ redundant nodes connected to the primary nodes of degree t, we are restricted to only $q^{k\pi_t}$ choices, while the rest contribute $q^{m-tk\pi_t}$ more choices.

Given any of the $q^{m-(t-1)k\pi_t}$ choices of r^m we can estimate $|\mathcal{G}_t(r^m)|$ from above by assuming that each primary node of degree d can take any of the $\lfloor d/t \rfloor$ label in $\mathcal{X}$ while still satisfying the t-wise coverage condition of Definition 1. This yields

$$|\mathcal{G}_t(r^m)| \leq \prod_{j=t}^{qt} \lfloor j/t \rfloor^{k\pi_j}, \tag{17}$$

and thus applying the union bound to (16), we get (14). □

For $t = 1, 2$ and $q = 2$, it is only necessary to evaluate (15) at three separate points (two of which are $\varepsilon = 1$ and 2, the third is anywhere inbetween) in order to show that the boundary of $\mathcal{R}_1$ or $\mathcal{R}_2$ from $\varepsilon = 1$ to 2 is linear. In particular, for $t = 2$, we can first choose $\varepsilon = 3/2$. No matter how we choose the values of π_2, π_3 and π_4, to satisfy (14) we must have $\rho \geq 1/2$.

Proof of Theorem 5: Achievability follows from Proposition 15. The converse is determined by evaluating 15. □

Remark 17: While the bound (15) *is tight for $t = 1$ and $t = 2$ when $q = 2$, it is not tight in general. It however allows us to make a general conclusion: since the bound is piecewise linear, it follows that the slope of $\mathcal{R}_t$ at the point* $(qt, 0)$ *of minimal redundancy is non-zero. It is also the best bound known to us for values of ε near qt.*

In the next section, we will discuss a bound that is better for ε away from q and when t is large. This converse outperforms the covering converse (Theorem 16) at certain ρ even for $q = 2$ and $t = 3$.

E. Ternary Alphabet and $t = 1$

Further progress on computing regions $\mathcal{R}_t$ for values of $q > 2$ seems to require finer arguments on graph structure. We can show the following result for $q = 3$ but the proof requires significant casework.

Theorem 18: For $q = 3$ and $t = 1$ we have

$$\mathcal{R}_1 = \{(\varepsilon, \rho) : \varepsilon \geq 3 - 2\rho, \varepsilon \geq 1, \rho \geq 0\} \tag{18}$$

and is achievable by the interpolation (11).

We give the proof in Appendix A.

V. Fundamental Limit for $t \to \infty$

Recall that as $t \to \infty$ the fundamental limit $\mathcal{R}_\infty \triangleq \limsup \mathcal{R}_t$ can be characterized as the set of wiring complexity-redundancy pairs, namely

$$\varepsilon = \frac{E}{kt}, \quad \rho = \frac{m}{kt} \tag{19}$$

over all values of t (see Proposition 4). The goal of this section is to prove the following result, that generalizes the binary version stated earlier in Theorem 6.

Theorem 19: Fix alphabet $|\mathcal{X}| = q$. The region $\mathcal{R}_\infty$ defined in (3) *is the closure of the set of points* (ε, ρ)*, parameterized by the distribution P_S on a finite support of $\mathbb{Z}_+$, and*

$$\varepsilon = \frac{\mathbb{E}[S]}{F(P_S)}, \quad \rho = \frac{1}{F(P_S)}, \tag{20}$$

$$F(P_S) \triangleq \min_{P_X} \max_{P_{Y|\underline{L}}} \min_{j \in [q]} \frac{1}{P_X(j)} \mathbb{E}[L_j \mathbb{1}\{Y = j\}] \tag{21}$$

where $\mathbb{E}[\cdot]$ is computed over random variables $S \in \mathbb{Z}_+, X \in [q], \underline{L} = (L_1, \ldots, L_q) \in \{0 \cup \mathbb{Z}_+\}^q, Y \in [q]$ with joint distribution

$$P_{S,\underline{L},Y}(s, \underline{\ell}, y) \triangleq P_S(s) P_{\underline{L}|S}(\underline{\ell}|s) P_{Y|\underline{L}}(y|\underline{\ell}). \tag{22}$$

where[7]

$$P_{\underline{L}|S}(\underline{\ell}|s) \triangleq \binom{s}{\ell_1, \cdots, \ell_q} \prod_{j=1}^{q} P_X(j)^{\ell_j}. \tag{23}$$

Theorem 19 gives Theorem 6 by substituting $P_{Y|\underline{L}}(0|(L_0, L_1))$ with $f(L_0, L_1)$, $P_X(0)$ with λ, and $P_X(1)$ with $1 - \lambda$. Also, the multinomial distribution is replaced by the binomial distribution.

We start the section by proving relevant properties of $F(P_S)$. We then use these properties to prove the achievability (i.e., upper bound) of Theorem 19. (This achievability proof explains why the quantities used in Theorem 19 are important.) Next, we present a symmetrization property which is the key idea of the converse argument of Theorem 19. Putting these elements together gives the complete proof.

Following the proof, we present a number of observations about Theorem 19. These include a section about how we compute Theorem 19 numerically and a section on the achievable region for designs where k is finite, but t and m are allowed to go to infinity. This result follows from the proof of Theorem 19. We also discuss how the Hamming block is optimal in this context.

A. Auxiliary Results About $F(P_S)$

Before proceeding further, we need to describe some technical properties of $F(P_S)$ and related quantities.[8]

[7] $P_{\underline{L}|S}$ is the multinomial distribution, Mult$(s, [P_X(1), \cdots, P_X(q)])$

[8] The notation $\frac{1}{k}\mathbb{Z}$ refers to the set of fractions with denominator k.

Definition 20 (Finitary F): We define $F_{k,n}(P_S)$ and $F_k(P_S)$ as follows:

$$F_{k,n}(P_S) \overset{\triangle}{=} \min_{P_X \in \frac{1}{k}\mathbb{Z}} \max_{P_{Y|\underline{L}^{(k)}} \in \frac{1}{n}\mathbb{Z}} \min_{j \in [q]} \frac{1}{P_X(j)} \mathbb{E}[L_j^{(k)} \mathbb{1}\{Y = j\}], \quad (24)$$

$$F_k(P_S) \overset{\triangle}{=} \min_{P_X \in \frac{1}{k}\mathbb{Z}} \max_{P_{Y|\underline{L}^{(k)}}} \min_{j \in [q]} \frac{1}{P_X(j)} \mathbb{E}[L_j^{(k)} \mathbb{1}\{Y = j\}], \quad (25)$$

where $\mathbb{E}[\cdot]$ is computed over random variables $S \in [k]$, $X \in [q], \underline{L}^{(k)} = (L_1, \ldots, L_q) \in \{0 \cup \mathbb{Z}_+\}^q, Y \in [q]$ with joint distribution

$$P_{S,\underline{L}^{(k)},Y}(s, \underline{\ell}, y) \overset{\triangle}{=} P_S(s) P_{\underline{L}^{(k)}|S}(\underline{\ell}|s) P_{Y|\underline{L}^{(k)}}(y|\underline{\ell}). \quad (26)$$

where[9]

$$P_{\underline{L}^{(k)}|S}(\underline{\ell}|s) \overset{\triangle}{=} \frac{\binom{kP_X(1)}{\ell_1} \cdots \binom{kP_X(j)}{\ell_j} \cdots \binom{kP_X(q)}{\ell_q}}{\binom{k}{s}}. \quad (27)$$

Note that the definition of $F_{k,n}$ is similar to that of $F(P_S)$, see (21), but with two changes: 1) values of P_X and $P_{Y|\underline{L}^{(k)}}$ (instead of $P_{Y|\underline{L}}$) are required to be integer multiples of $\frac{1}{k}$ and $\frac{1}{n}$, respectively; and b) $P_{\underline{L}^{(k)}|S}$ is (multivariate) hypergeometric, instead of multinomial. The function $F(P_S)$ which we are ultimately interested in for this section is the limit of $F_k(P_S)$ for $k \to \infty$, which itself is a limit of $F_{k,n}(P_S)$. The function $F_k(P_S)$ is an important quantity which bounds the rate region for designs with finite k, which we will discuss later in V-F.

Proposition 21: For any P_S with finite expectation we have

$$F_k(P_S) - \frac{\mathbb{E}[S]}{n} \leq F_{k,n}(P_S) \leq F_k(P_S). \quad (28)$$

Also, there exists a sequence $\epsilon_k \to 0$ such that for any P_S on $\mathbb{Z}_+$ with finite third moment we have

$$|F_k(P_S) - F(P_S)| \leq \frac{\mathbb{E}[S^3]}{2k} + \epsilon_k. \quad (29)$$

See Appendix B for proofs.

B. Subset Design Achievability and Upper Bound

The next proposition gives bounds on the performance of subset designs in terms of $F_{k,n}(P_S)$ and $F_k(P_S)$.[10]

Proposition 22: Let $q = |\mathcal{X}|$ and fix $k \in \mathbb{Z}$. Let $G = \bigvee_{i=1}^n G'$, where G' is a subset design with $P_S(s)$ as the proportion of redundant nodes with degree s for $s \in [k]$. If G is a $(k, m, t, E)_q$-design, where $E = m\mathbb{E}[S]$ and t is the maximum number of defects G can correct, then

$$\frac{m}{k} F_{k,n}(P_S) \leq t \leq \frac{m}{k} F_k(P_S). \quad (30)$$

Proof: First we show the upper bound that $t \leq \frac{m}{k} F_k(P_S)$.

Consider any labeling $w^k \in \mathcal{X}^k$ of the k primary nodes of G. Let the frequency which each label occurs in the labeling have empirical distribution P_X (that is, if k_i of the k primary nodes have label i, then $P_X(i) = \frac{k_i}{k}$). Given this labeling, we define the *type* of each redundant node v to be $\underline{\ell} = (\ell_1, \cdots, \ell_q)$, where ℓ_j is the number of primary nodes with label j which is a neighbor of redundant node v. (If the degree of the redundant node is s, then $\sum_{i=1}^q \ell_i = s$.) Because G is a subset design, the proportion of degree s redundant nodes in G with type $\underline{\ell}$ is

$$P_{\underline{L}^{(k)}|S}(\underline{\ell}|s) = \frac{\binom{kP_X(1)}{\ell_1} \cdots \binom{kP_X(q)}{\ell_q}}{\binom{k}{s}}. \quad (31)$$

Now, for any choice of labeling $r^m \in \mathcal{X}^m$ of the m redundant nodes, let $P_{Y|\underline{L}^{(k)}}(j|\underline{\ell})$ represent the proportion (empirical distribution) of redundant nodes of type $\underline{\ell}$ which are labeled j. For each label j, we can count the *average* number of matching redundant node neighbors a primary node u with label j has by summing up all the edges between primary and redundant nodes both with label j, and then dividing this by the total number of primary nodes with label j. This average is

$$\tilde{t}(j) \overset{\triangle}{=} \frac{1}{kP_X(j)} \sum_s m P_S(s) \sum_{\underline{\ell}} P_{\underline{L}^{(k)}|S}(\underline{\ell}|s) \ell_j P_{Y|\underline{L}^{(k)}}(j|\underline{\ell}) \quad (32)$$

$$= \frac{m}{k} \frac{1}{P_X(j)} \mathbb{E}[L_j \mathbb{1}\{Y = j\}]. \quad (33)$$

The label j where this average is lowest determines the upper bound on the number of defects G with labeling w^k and r^m can correct. This upper bound is given by $\min_j \tilde{t}(j)$. We have the freedom to pick the redundant node labeling r^m with the empirical distribution $P_{Y|\underline{L}^{(k)}}$ which maximizes the average. The defect correcting number needs to hold for all possible w^k, so the empirical distribution P_X which gives the lowest value of $\max_{P_{Y|\underline{L}^{(k)}}} \min_j \tilde{t}(j)$ determines t. This gives the upper bound on t.

We now show the lower bound $\frac{m}{k} F_{k,n}(P_S) \leq t$.

Given any labeling $w^k \in \mathcal{X}^k$ of the primary nodes with empirical distribution P_X, let

$$P_{Y_n|\underline{L}^{(k)}} = \operatorname*{argmax}_{P_{Y|\underline{L}^{(k)}} \in \frac{1}{n}\mathbb{Z}} \min_{j \in [q]} \frac{1}{P_X(j)} \mathbb{E}[L_j \mathbb{1}\{Y = j\}]. \quad (34)$$

For each $\underline{\ell}$, $P_{Y_n|\underline{L}^{(k)}}(j|\underline{\ell}) = \frac{c_j}{n}$ for some $c_j \in \mathbb{Z}_+ \cup 0$, and $\sum_j c_j = n$. Because G is a merging of n copies of G', we can partition the copies of G' in G into sets of size $c_1, \ldots, c_q$. The jth set is a set of c_j copies of G'. Label all redundant nodes of type $\underline{\ell}$ in the jth set the value j. We can determine that each primary node u with label j has a total of $P_S(s) m \frac{\ell_j}{P_X(j)k} P_{\underline{L}^{(k)}|S}(\underline{\ell}|s)$ redundant nodes of type $\underline{\ell}$ in its neighborhood. This redundant node labeling scheme assigns exactly $P_{Y_n|\underline{L}^{(k)}}(j|\underline{\ell})$ of these neighbors the label j.

Repeat this labeling process for each redundant node type $\underline{\ell}$. Summing over all $\underline{\ell}$ and all s will get that the total number of redundant nodes with label j in the neighborhood of primary node u is $\sum_{s,\underline{\ell}} P_S(s) m \frac{\ell_j}{P_X(j)k} P_{\underline{L}^{(k)}|S}(\underline{\ell}|s) P_{Y_n|\underline{L}^{(k)}}(j|\underline{\ell})$.

[9] $P_{\underline{L}^{(k)}|S}$ is the multivariate hypergeometric distribution, HyperGeom$(s, k, [P_X(1), \cdots, P_X(q)])$

[10] This proposition initially used random coding as an argument. Random coding has since been replaced.

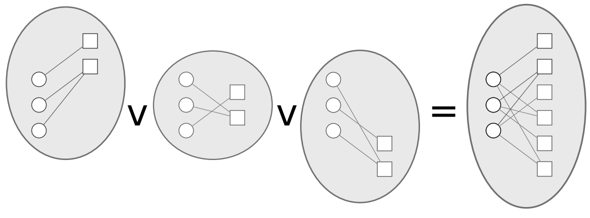


Fig. 9. Example merging permutations of the same design. The resulting design is a subset design. In this example, for clarity, we did not show all 6 possible permutations of the 3 primary nodes in the original design. The 3 distinct permutations shown was enough to create a subset design.

Using this scheme, G can correct at least

$$t \geq \min_{P_X \in \frac{1}{k}\mathbb{Z}} \min_{j \in [q]} \sum_{s,\underline{\ell}} P_S(s) m \frac{\ell_j}{P_X(j)k} P_{\underline{L}^{(k)}|S}(\underline{\ell}|s) P_{Y_n|\underline{L}^{(k)}}(j|\underline{\ell}) \tag{35}$$

$$= \frac{m}{k} \min_{P_X \in \frac{1}{k}\mathbb{Z}} \max_{P_{Y|\underline{L}^{(k)}} \in \frac{1}{n}\mathbb{Z}} \min_{j \in [q]} \frac{1}{P_X(j)} \mathbb{E}\left[L_j \mathbb{1}\{Y = j\}\right] \tag{36}$$

$$= \frac{m}{k} F_{k,n}(P_S) \tag{37}$$

defects. □

C. Converse and Proof of Theorem 19

The converse needed to show Theorem 19 is surprisingly simple. The main idea is the following:

Proposition 23 (Symmetrization): If there exists a $(k, m, t, E)_q$-design then there exists a permutation-invariant $(k, m \cdot k!, t \cdot k!, E \cdot k!)_q$-design.

Proof: Let G be a $(k, m, t, E)_q$-design. We will merge G exactly $k!$ number of times. The key is that each copy will be merged by identifying with a permutation of the original primary nodes.

Start with an ordering of the primary nodes in the design G. For each $\sigma \in S_k$ (the full symmetric group of k elements), let G_σ be isomorphic to the design G, with the order of its primary nodes transformed by σ. Then merge G_σ for all $\sigma \in S_k$ identifying primary nodes in the same order.

Let the result be

$$G_{\text{PERM}} = \bigvee_{\sigma \in S_k} G_\sigma. \tag{38}$$

G_{PERM} is constructed to be permutation invariant. (For any redundant node v in G, if v has degree s, every set of s nodes in G_{PERM} needs to be connected together by a copy of v.) By Proposition 14 G_{PERM} is a $(k, m \cdot k!, t \cdot k!, E \cdot k!)_q$-design. □

See Fig. 9 for an example of merging permutations to obtain a subset design.

In view of Propositions 10 and 23, we see that in terms of the values $\frac{E}{kt}, \frac{m}{kt}$ every design on k primary nodes is at most as good as a subset design on k primary nodes (meaning the pair of values an arbitrary design will achieve has the same or a worse trade-off than what a subset design can achieve). Performance of the latter is completely characterized by Proposition 22. Now we can combine the results to prove Theorem 19.

Proof: Proof of Theorem 19

Achievability: Fix $P_S \in \mathbb{Q}$ with finite support. For each k and n, it is always possible to construct a subset design G' on k primary nodes where the proportion of redundant nodes of degree s are given by $P_S(s)$. Let $G = \bigvee_{i=1}^n G'$ so that by Proposition 22, subset design G is a $(k, m, t, E)_q$-design so that $\frac{tk}{m} \geq F_{k,n}(P_S)$ and $E = m\mathbb{E}[S]$. Since $F_{k,n}(P_S) \to F(P_S)$, there must exist a sequence of subset designs G_i which are $(k_i, m_i, t_i, E_i)_q$-designs where

$$\varepsilon = \frac{E_i}{k_i t_i} \to \frac{\mathbb{E}[S]}{F(P_S)}, \rho = \frac{m_i}{k_i t_i} \to \frac{1}{F(P_S)}. \tag{39}$$

Thus

$$\left(\frac{\mathbb{E}[S]}{F(P_S)}, \frac{1}{F(P_S)}\right) \in \mathcal{R}_\infty. \tag{40}$$

Since $F(P_S)$ is continuous in P_S, (40) holds for any P_S with finite support.

Converse: For any design G which is a $(k, m, t, E)_q$-design, there exists a subset design G' which is a $(k, m{\cdot}k!, t{\cdot}k!, E{\cdot}k!)_q$-design by Proposition 23. Let P_S be so that $P_S(s)$ represents the proportion of redundant nodes in G' with degree s. Then $E = m\mathbb{E}[S]$. Let t' be the number of defects G' can correct. Using Proposition 22 and $F_k(P_S) \leq F(P_S)$ (cf. Lemma 32 in Appendix C),

$$t \cdot k! \leq t' \leq \frac{m \cdot k!}{k} F_k(P_S) \leq \frac{m \cdot k!}{k} F(P_S). \tag{41}$$

Then for design G, $\varepsilon = \frac{E}{tk} \geq \frac{\mathbb{E}[S]}{F(P_S)}$ and $\rho = \frac{m}{tk} \geq \frac{1}{F(P_S)}$. Thus, the limit of $(\frac{E_i}{k_i t_i}, \frac{m_i}{k_i t_i})$ for any sequence of $(k_i, m_i, t_i, E_i)_q$-designs must be in the closure of $(\frac{\mathbb{E}[S]}{F(P_S)}, \frac{1}{F(P_S)})$ for all P_S with a finite support. □

D. Observations About Theorem 19

1) Threshold Solution: The optimal value for $P_{Y|\underline{L}}$ tells us what the optimal labeling of redundant nodes should be. It turns out that for most values of $\underline{\ell}$, $P_{Y|\underline{L}}(j|\underline{\ell})$ is either 0 or 1.

For an illustration of this, consider the binary alphabet (or $q = 2$) case and the design $S(k, s)$. The types are $\underline{\ell} = (\ell_0, \ell_1)$. Given any empirical distribution P_X of the primary node labels, the optimal labeling of the redundant nodes must be so that redundant nodes with larger values of ℓ_1 are assigned the label 1 instead of redundant nodes with smaller values of ℓ_1. Otherwise, we can always swap the labelings and increase the number of defects corrected. In fact, even when there are multiple subset sizes, it is possible to find an optimal solution where the value of $P_{Y|\underline{L}}$ depends only on the ratio of ℓ_0 to ℓ_1.

Proposition 24: For $\mathcal{X} = \{0, 1\}$, the solutions $P_{Y|\underline{L}}$ which attain the maximum in (21) *must have the following form*

$$P_{Y|\underline{L}}(0|\underline{\ell}) = \begin{cases} 1 & \textit{if } \frac{\ell_0}{\ell_0+\ell_1} > \gamma \\ 0 & \textit{if } \frac{\ell_0}{\ell_0+\ell_1} < \gamma \\ \mu(\ell_0+\ell_1) & \textit{if } \frac{\ell_0}{\ell_0+\ell_1} = \gamma \end{cases} \tag{42}$$

where $\gamma \in [0, 1]$ and $\mu(s) \in [0, 1]$ for each $s \in \mathbb{Z}_+$.[11]

[11] There is not necessarily a unique solution for $\mu(s)$. One such solution has $\mu(s_a) = \mu(s_b)$ for all s_a, s_b.

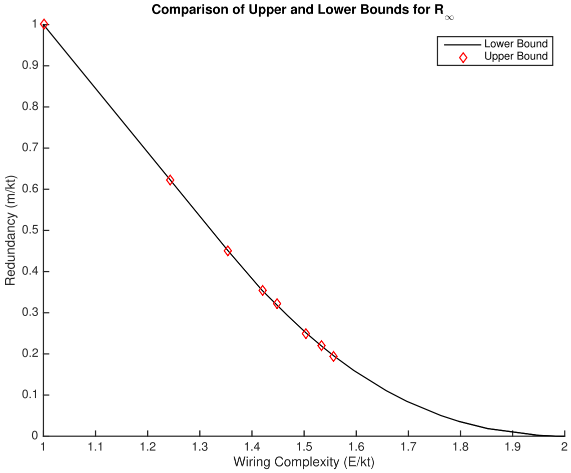


Fig. 10. Approximate converse bound compared with achievable points for $q = 2$.

(See Appendix D for proof.) Generalizing to larger alphabet sizes, the space of all possible $\underline{\ell}$ will be partitioned into q pieces depending on the relative ratios of values in $\underline{\ell}$. The interior of each piece will have all types assigned to the same label, that is $P_{Y|L}(j|\underline{\ell}) = 1$ for some j. The values of $\underline{\ell}$ on the boundary may be split between 2 or more values.

Notice that in light of Theorem 19 and Proposition 24, computing the optimal redundant node labeling for subset designs given a fixed primary node labeling is easy. For general designs, this is NP-Hard.

2) Worst-Case P_X: The worst-case distribution of primary node labels which gives the result in Theorem 19 is not obvious, even in the binary alphabet case. When $\mathcal{X} = \{0, 1\}$, we can easily determine that for subset designs $S(k, s)$ with even s, the worst-case P_X is when $P_X(0) = P_X(1) = \frac{1}{2}$. However, when s is odd, this is not true. When $s = 3$, the worst-case P_X is determined by a solution to a cubic polynomial. When a merging of different subset designs are used or a larger alphabet is used, it is unclear how to find the worst-case P_X analytically. This makes finding the worst-case P_X the main difficulty in evaluating the optimization equation in Theorem 19 for given values of P_S. (The equation is non-convex in P_X.)

E. Numerical Upper and Lower Bounds

Since the optimization presented in Theorem 6 is difficult to evaluate exactly, instead, we give an approximation for the boundary by establishing computable almost tight upper and lower bounds for when $q = 2$. The details can be found in Appendix E and a comparison is presented on Fig. 10. As can be seen, the gap between the bounds is on the order of 10^{-3} and virtually indistinguishable on the plot. The best known achievable point in $\mathcal{R}_\infty$ for selected fixed values of $\mathbb{E}[S]$ are given in Table I. These points are found by searching and using weights from the converse bound method presented in Appendix E.

We observed the following effects about designs near the boundary of $\mathcal{R}_\infty$ while experimenting with Theorem 6:

- The design has 4 or 5 distinct subset sizes
- Odd number subset sizes are more common
- The subset sizes which make up most of the design are consecutive, possibly skipping even subset sizes

TABLE I

ACHIEVABLE POINTS

$\mathbb{E}[S]$	Support set with corresponding P_S	Point (ε, ρ) in $\mathcal{R}_\infty$
2	[1,3,4,5] , [0.62, 0.21, 0.10, 0.07]	(1.24, 0.61)
3	[1,3,4,5] , [0.24, 0.41, 0.20, 0.14]	(1.35, 0.45)
4	[3,4,5,6,7] , [0.52, 0.21, 0.13, 0.02, 0.12]	(1.42, 0.35)
5	[3,4,5,7] , [0.31, 0.23, 0.28, 0.18]	(1.40, 0.29)
6	[5,6,7,9] , [0.45, 0.31, 0.14, 0.10]	(1.47, 0.29)
7	[5,6,7,8,9] , [0.35, 0.01, 0.13, 0.32, 0.19]	(1.53, 0.22)
8	[7,8,9,11] , [0.40, 0.36, 0.16, 0.08]	(1.56, 0.19)

F. Results for Finite k

To develop the proof for Theorem 19, we showed intermediate results on designs with k primary nodes and observed what occurs when $k \to \infty$. We can use these intermediate results to determine the achievable regions for designs on k primary nodes.

Definition 25: For fixed q and $k \in \mathbb{Z}_+$, we define the region $\mathcal{R}^k_\infty$ as the closure of the set of all achievable pairs $(\frac{E}{kt}, \frac{m}{kt})$:

$$\mathcal{R}^k_\infty \triangleq \text{closure}\left\{\left(\frac{E}{kt}, \frac{m}{kt}\right) : \exists (k, m, t, E)_q\text{-}design\right\}. \quad (43)$$

Similar to regions $\mathcal{R}_t$ and $\mathcal{R}_\infty$, the region $\mathcal{R}^k_\infty$ is convex. We can apply the proof for Claim 5 in Proposition 4 replacing the expression (10) with

$$G = \left(\bigvee_{i=1}^{pt_2} G_1\right) \vee \left(\bigvee_{i=1}^{(q-p)t_1} G_2\right) \quad (44)$$

to show this.

Claims 1 and Claim 2 of Proposition 4 also hold for $\mathcal{R}^k_\infty$.

Theorem 26: Fix alphabet $|\mathcal{X}| = q$. The region $\mathcal{R}^k_\infty$ defined in (43) *is the closure of the set of points (ε, ρ), parameterized by the distribution P_S on $[k]$, where*

$$\varepsilon = \frac{\mathbb{E}[S]}{F_k(P_S)}, \quad \rho = \frac{1}{F_k(P_S)}, \quad (45)$$

and $F_k(\cdot)$ is defined in (25).

Proof: The achievability and converse of this theorem follows from Proposition 22 (with $F_k(P_S)$ is continuous in P_S) and Proposition 23 respectively. □

Using (45), we can plot the achievable region $\mathcal{R}^3_\infty$ when $q = 2$ (see Fig. 11). The most salient aspect of $\mathcal{R}^3_\infty$ is that the point achieved by the Hamming block (see Fig. 5(a)) is a corner point of this region. It is the only corner other than the usual corner point $(1, 1)$ achieved by the repetition design.

Corollary 27 (Hamming Block Corner Point): The value given by the Hamming block is a corner point of $\mathcal{R}^3_\infty$ for $\mathcal{X} = \{0, 1\}$.

The proof for Corollary 27 and the methods used to calculate $\mathcal{R}^3_\infty$ are in Appendix F. The implication of this result is that for any design on $k = 3$ primary nodes, no design

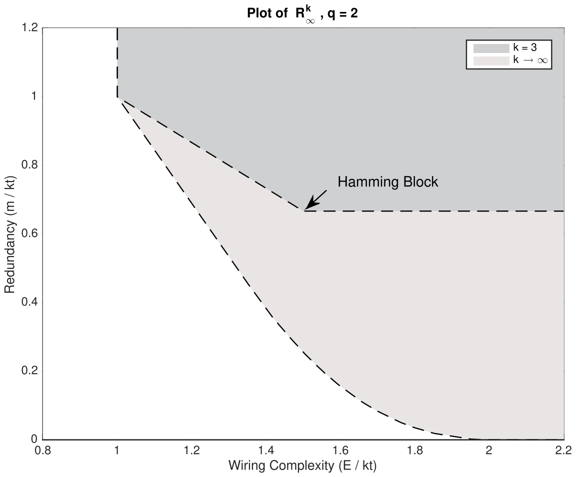


Fig. 11. Region $\mathcal{R}^3_\infty$ (see (43)) compared with $\mathcal{R}_\infty$ for $q = 2$.

has a better trade-off between redundancy $\frac{m}{kt}$ and wiring complexity $\frac{E}{kt}$ than the Hamming block, even if we allow the design to have arbitrarily many edges and redundant nodes.

VI. Discussion

We conclude with a discussion of some implications of our results, some extensions and future work.

A. Implications on Practical Designs

The result for $\mathcal{R}_1$ and $\mathcal{R}_2$ (for $q = 2$) demonstrates that for correcting small defects, the best solution in the limit of a large number of primary nodes is a linear combination of two basic designs, the repetition design and the complete design. (Though this design is not optimal for finite k. Slight improvements can be made by removing a few edges.)

Theorem 19 gives a result for asymptotic t, and while practically no application is going to need to correct asymptotically many defects, the region defined by the result gives a converse bound for $\mathcal{R}_t$ for all finite t by virtue of Claim 4 from Proposition 4. All regions $\mathcal{R}_t$ must lie between $\mathcal{R}_1$ and $\mathcal{R}_\infty$, approaching the latter as $t \to \infty$. Hence Theorem 19 describes the fundamental limit for the trade-off between redundancy and wiring complexity.

The numerical results for asymptotic t and $q = 2$ imply that the designs which are close to optimal for large t use redundant nodes with a limited set of degrees. The best achievable points found for $\mathcal{R}_\infty$ for fixed values of $\mathbb{E}[S]$ each use redundant nodes with degrees within 2 or 3 values of $\mathbb{E}[S]$.

Results for $\mathcal{R}^k_\infty$ define what is optimal for finite k in terms of the number of defects correctable per use of redundancy and edges. We know exactly what this region looks like for $k = 3$ and can determine that the Hamming block is in fact the optimal design. Evaluating $\mathcal{R}^k_\infty$ for larger values of k gives exactly what trade-offs are realizable.

Also note that in the asymptotic t results, the optimal trade-off is obtainable by designs which has regular primary node degree (since subset designs are always regular). Not only that, but finding the best labeling of redundant nodes for subset desgins corresponds to finding $P_{Y|\underline{L}}$ in the statement of Theorem 19, which is easy to compute.

B. Comparison to Other Models for Defect Tolerance

This paper studies the defect-tolerance model where steps proceed as follows:

a. bipartite graph (interconnect) is designed;
b. primary nodes get q-ary labeling;
c. redundant nodes are assigned q-ary labels (so that each primary node has t neighbors with matching label).

There are two natural variations where the sequence of steps are interchanged:

- *adaptive graph*: b.→a.→c.
- *non-adaptive redundancy*: a.→c.→b.

In the first case, the design of the edges of the graph is a function of the q-ary labels, while in the second case the redundant nodes are not allowed to depend on the labeling of primary nodes.

It is clear that the setting considered in this paper (a.→b.→c.) is an intermediate case. That is, any t-defect correcting design in the sense of Definition 1 is also t-defect correcting in the sense of the *adaptive graph*. Similarly every design with *non-adaptive redundancy* should work in the sense of Definition 1.

The fundamental redundancy-wiring complexity trade-off is defined similarly to (2). However, for both cases it is rather easy to determine this trade-off for any $t \geq 1$:

- *adaptive graph*: Clearly the number of edges $E \geq kt$. This can be attained with (asymptotically) zero-redundancy by adding t redundant nodes of each label (for a total of $m = qt$) and connecting every primary node only to relevant t redundant nodes. Consequently, here
$$\mathcal{R}_t = \{(\varepsilon, \rho) : \varepsilon \geq 1, \rho \geq 0\}. \tag{46}$$
- *non-adaptive redundancy*: Again, clearly the number of edges is $E \geq qkt$. This can be attained with (asymptotically) zero-redundancy by adding t redundant nodes of each label (for a total of $m = qt$) and connecting every primary nodes to all of qt redundant ones (i.e., using $K(k, qt)$ design). Consequently
$$\mathcal{R}_t = \{(\varepsilon, \rho) : \varepsilon \geq q, \rho \geq 0\}. \tag{47}$$

These observations are summarized in Fig. 12 for $q = 2$.

C. Relation to (t, t)-Colorable Hypergraphs

There is a purely graph-theoretic way to look at our problem. For this we bring up the concept of a (t, t)-graph coloring introduced in [20]. A hypergraph is called (t, t)-colorable if for every $\{0, 1\}$-coloring of hyperedges there exists a $\{0, 1\}$-coloring of vertices so that each edge contains t vertices of its color. Define

$$\begin{aligned} d_t(k, m) = \min(&\text{average edge-size:} \\ &\text{all } (t, t) \text{ colorable hypergraphs on} \\ &m \text{ vertices and } k \text{ hyperedges}). \end{aligned} \tag{48}$$

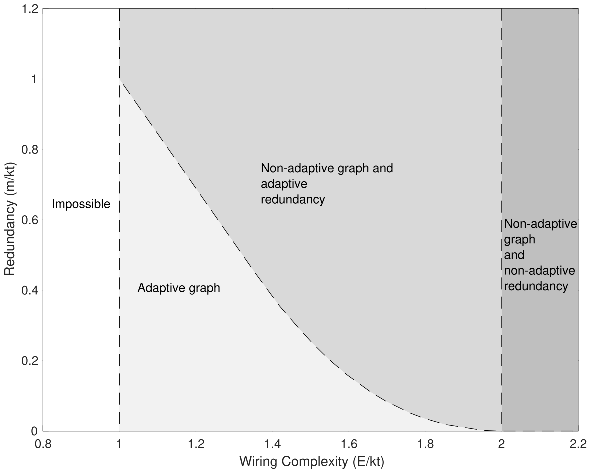


Fig. 12. Comparison of redundancy-wiring complexity trade-offs for different levels of adaptivity for $q = 2$.

It is not hard to see that our problem with binary $\mathcal{X}$ and (t, t)-coloring are one-to-one related: the vertices correspond to primary nodes and the hyperedges correspond to redundant nodes. More precisely we have

Proposition 28: Fix binary $\mathcal{X}$. The boundary of $\mathcal{R}_t$ is given by

$$\liminf_{k\to\infty} \frac{1}{t} d_t(k, \lceil \rho k t \rceil). \tag{49}$$

The boundary of $\mathcal{R}_\infty$ is given by

$$\liminf_{t\to\infty} \liminf_{k\to\infty} \frac{1}{t} d_t(k, \lceil \rho k t \rceil). \tag{50}$$

Proof: Note that for a fixed t, if for some pair (ε, ρ) we have $\frac{1}{t} d_t(k, \lceil \rho k t \rceil) = \varepsilon$ for some k, then by copying (Proposition 13), there exists infinitely many values of $k' > k$, where $\frac{1}{t} d_t(k', \lceil \rho k' t \rceil) \leq \varepsilon$. It follows from Proposition 4 that $\liminf_{k\to\infty} \frac{1}{t} d_t(k, \lceil \rho k t \rceil)$ must correspond to the boundary of $\mathcal{R}_t$.

Similarly, by merging (Proposition 14), any pair (ε, ρ) where there is some t such that

$$\liminf_{k\to\infty} \frac{1}{t} d_t(k, \lceil \rho k t \rceil) = \varepsilon$$

for some t, must also have infinitely many values of $t' > t$ where $\liminf_{k\to\infty} \frac{1}{t'} d_{t'}(k, \lceil \rho k t' \rceil) \leq \varepsilon$. □

Hypergraphs were used [20] to show a specific achievability scheme for storing data with bitprobes. This achievability scheme thresholds of the number of neighbors with value 0 to determine values of data points. Using the connection our defect correcting designs have with hypergraphs, we can use Theorem 6 to show a converse bound on the size of the encoded vector for bitprobes that use thresholding. However, the constants we get from applying our work to bitprobes does not do better than those cited in [20]. For instance, for 3 bitprobes and vectors where at most 1/3 of the entries are 1, our result gives that the ratio of the length of the encoded vector to the length of the original vector must be greater than .21 whereas the method cited in [20] gives that the ratio must be greater than .48.

D. Stochastic Defects

This work considered correcting arbitrary (worst-case) defect patterns. Suppose that instead we are interested in correcting fraction α of defects (i.e., $t = \alpha\ (k + m)$) on k primary and m redundant nodes. In this scenario, the number of redundant nodes m would need to grow as a function of k in order to keep up with the number of defects needed to be corrected. If α is too large, it is not possible to find designs which corrects $\alpha(k + m)$ defects for arbitrarily large k.

To see this, note that correcting worst case t defects with alphabet size q requires at least qt redundant nodes.

$$m \geq qt \tag{51}$$

$$m \geq q\alpha(k + m) \tag{52}$$

$$m(1 - q\alpha) \geq q\alpha k. \tag{53}$$

The quantity on the right-hand side of (53) needs to be positive, so it must be that $\alpha < \frac{1}{q}$.

Additionally, the only designs which can correct fraction $\alpha < \frac{1}{q}$ of defects for growing k are designs with the same redundancy and wiring complexity as complete designs. From our results in Theorem 19, we know that there exists $(k, m, t, E)_q$-designs so that

$$\frac{m}{kt} \to c \tag{54}$$

for some constant c. When $t = \alpha(k + m)$,

$$\frac{m}{k\alpha(k + m)} > c \tag{55}$$

$$m(1 - ck\alpha)\ >\ ck^2\alpha. \tag{56}$$

In order for (56) to hold, the right-hand side must be positive, so it must be that $c \to 0$ as k becomes arbitrarily large. The point in $\mathcal{R}_\infty$ where $\frac{m}{kt} \to 0$ corresponds to the complete design.

In light of these results, it is natural to ask what happens if instead we relaxed the requirement to correcting i.i.d. Bernoulli(α) defects in the sense of high probability (computed over distribution of defects and primary assignments). It turns out that in such probabilistic model, correcting fraction-α of defects is possible with designs possessing $O(k \log k)$ edges and $O(k)$ redundant nodes. See [21, Sec. 4.4, Ths. 4.10 and 4.15] for more (pp. 63-66).

E. Future Work

One direction for future work involves extensions beyond the bipartite graph. We chose to study the one-level bipartite graph model for simplicity, but experiments like Teramac [19] have demonstrated the effectiveness of multi-level hierarchical designs. This leads to the question of what are the optimal trade-offs when hierarchical models of redundancy are used. The hierarchical model would include intermediate nodes which can facilitate connections of edges. The presence of the intermediate nodes can greatly reduce the number of edges. To correct t defects, we can connect each primary node to t intermediate nodes. Regardless of the number of primary nodes, the intermediate nodes can connect to finitely many redundant nodes. This way, we are able to achieve a wiring complexity of t and redundancy of 0 (asymptotically). In such a case, we would be interested in finding the fundamental trade-offs with the number of intermediate nodes as a parameter.

F. Open Problems

Regions which are still to be determined include:

- $\mathcal{R}_t$ for $t > 2$ and $q = 2$
- $\mathcal{R}_t$ for $t > 1$ and $q \geq 3$

For $q = 2$, it is also unknown what the smallest value of t is for which $\mathcal{R}_t$ does not equal the region defined in (5).

Appendix

A. Proof of Theorem 18

Proof: Define $\mathcal{R}_1$ as in (18). We will show that all $(k, m, 1, E)_3$-designs must lie in $\mathcal{R}_1$. Let the primary nodes have labels in $\mathcal{X} = \{0, 1, 2\}$.

Instead of saying that a given bipartite graph is 1-defect correcting for alphabet of size $q = 3$, we will say (for brevity) that a graph satisfies property (*).

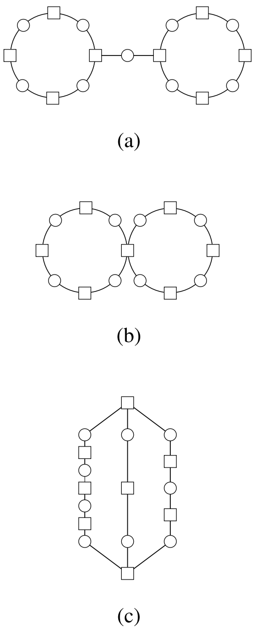
(*) is the property that for any labeling of the primary nodes in $\mathcal{X}^k$, where k is the number of primary nodes, there exists a labeling of the redundant nodes so that each primary node has at least one redundant node neighbor with the same labeling.

The steps for this proof are:

1) Show that designs with primary nodes of degree 3 and greater can be disregarded.
2) Show that in order to satisfy (*), designs with any primary nodes of degree 1 must be in $\mathcal{R}_1$.
3) Show that in order to satisfy (*), designs with primary nodes all of degree 2 must be in $\mathcal{R}_1$.
   a) Show designs containing two disjoint cycles connected by a path (see Fig. 13(a)) violate (*)
   b) Show designs containing two cycles which intersect at one point (see Fig. 13(b)) violate (*)
   c) Show designs containing two cycles which intersect at multiple points (see Fig. 13(c)) violate (*)

Step (1): The key to this step is to make a graph with (almost) equivalent parameters where nodes of degree 3 or more are in a separate component. For any $(k, m, 1, E)_3$-design G, define a new design G' with the same number of primary nodes k and the number of redundant nodes as $m' = m + qt = m + 3$. The added redundant nodes are connected to each of the primary nodes that have degree (in G) larger or equal to 3. The remaining primary nodes are connected in G' exactly as in G. It is clear that G' still satisfies (*), has the same (or smaller) number of edges and (asymptotically in m) the same redundancy ρ. This shows, that without loss of generality we can assume that there are no primary nodes of degree greater than 3 and all nodes of degree 3 form a complete bipartite graph disjoint from the rest of the design. We can ignore this separate component.

Step (2): The main argument of this step is to show that if the design has any primary node of degree 1, the design must be a tree.


(a)

(b)

(c)

Fig. 13. Example designs for the different cases considered in the proof of Theorem 18. (a) Case (3a). (b) Case (3b). (c) Case (3c).

We will say a primary node is *adjacent* to another primary node if the two primary nodes share a redundant node as a common neighbor.

Suppose the design has a primary node u_0 of degree 1 and no primary nodes of degree 3 or more. For all primary and redundant nodes, we will consider the node's shortest distance to u_0. If a node is distance i from u_0, we say that the node is at level i. Since the design is bipartite, even levels have primary nodes, and odd levels have redundant nodes. Let tier n mean the levels $2n$ and $2n + 1$.

Consider the labeling of the primary nodes where all primary nodes in even tiers are labeled 0's and all primary nodes in odd tiers are labeled 1's. Since u_0 only has one neighbor, in order to satisfy (*), we must label the one redundant node in tier 0 the value 0. The primary nodes in tier 1 each have the redundant node in tier 0 and some redundant node in tier 1 as neighbors. The redundant node in tier 0 is already labeled a 0, so all redundant nodes in tier 1 must be labeled 1's in order to satisfy (*). Since primary nodes in tier 2 are labeled with 0's, then all redundant nodes in tier 2 must also be labeled 0's. Continuing this argument by induction, all redundant nodes in a tier must be labeled the same value as the primary nodes in that tier. If the design is a tree, then this labeling scheme satisfies (*).

Now suppose the design has cycles. Find the lowest tiered primary node which completes a cycle, that is the lowest tiered primary node u_c in tier c which has one redundant node in tier $c - 1$ and the other redundant node it has is shared by a primary node in tier b, where $b \leq c$. (It could be that two nodes in tier c share the same redundant node in tier c. Pick either as u_c). Now switch the label of u_c to 2 and keep all the other labels the same. Both redundant neighbors of u_c must be labeled a 0 or 1, so we do not satisfy (*).

Condition (*) is not satisfied unless the design has no cycles and is a tree. If the design is a tree, it must have at least the same number of redundant nodes as primary nodes, so the design lies in $\mathcal{R}_1$. We can now assume that all primary nodes are of degree 2.

Notice having $q \geq 3$ is important to avoid existence of even cycles.

Step (3): Our goal is to prove that if all primary nodes have degree 2, then $\frac{m}{k} \geq \frac{1}{2}$. We will instead prove something stronger: For $k > 4$ we must have $m \geq k$. For $k = 4$ we must have $m \geq 3$.

Because of copying (see (8)-(9)), it is sufficient to prove the above for designs on a single component. If there is a redundant node with degree 1, we can remove this redundant node with its neighboring primary node from the design and make it a separate component. We can assume all redundant nodes must also have degree 2 or more.

We will call a labeling of primary nodes *alternating* if adjacent primary nodes have different labels.

Lemma 29: If a design with k primary nodes, all of degree 2*, and* $k-1$ *or fewer redundant nodes, all of degree 2 or 3, can be labeled alternatingly, then the design cannot satisfy (*).*

Proof: If there is an alternating labeling, at most each redundant node can only match the labeling of one of its neighboring primary nodes. There can only be at most $k-1$ matches, so there exists one primary node which does not have a neighboring redundant node with the same label as itself. □

Suppose a design with all primary nodes of degree 2 is such that $m < k$. Then, some redundant node in the design must have degree 3 or more. Pick the separate component with this redundant node, and let A be a cycle in this separate component (if this component does not have cycles, then $m \geq k$ as in Step 2). In order for cycle A to be in this component, a redundant node with degree more than 2 must also be in A. Call this redundant node v_0. Call the neighbor of v_0 which is not in A u_0.

Build a path B in the design starting at primary node u_0 as follows: The second node in path B will be the neighbor of u_0 which is not v_0. We can pick the next node in the path arbitrarily. The path ends when we reach a node in A or a node already in B. To show that (*) does not hold on the design, it is sufficient to show that (*) is not satisfied on the subgraph $A \cup B$.

Depending on the endpoint of path B, we have several cases:

Case (3a): Endpoint of B coincides with an intermediate point of B (see Fig. 13(a))

Let v_1 be the redundant node in path B where the path B ends. Rename the cycle created by path B to cycle C. The subgraph $A \cup B \cup C$ satisfies the conditions of Lemma 29 so we need only show that we can find an alternating label. The nodes v_0 and v_1 are the only redundant nodes with degree 3. It is clear that by starting with a alternating label of the neighbors of v_0, we can find an alternating label for the path between v_0 and v_1, and then an alternating label of the neighbors of v_2. After this, it is easy to find an alternating label for the rest of A and C.

Case (3b): Endpoint of B is node v_0 (see Fig. 13(b))

Let cycle C be the cycle formed using path B and v_0. As long as one of cycle A and cycle C have more than 2 primary nodes, the design violates (*).

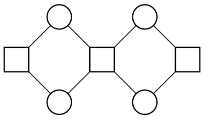


Fig. 14. Design which satisfies (*). Exception to case (3b).

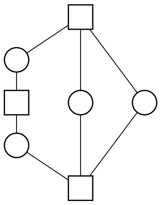


Fig. 15. Design which satisfies (*). Exception to case (3c).

Consider when the labeling is so that the two primary nodes in the larger cycle (assume this to be cycle A) neighboring v_0 are labeled the same value, say 0. The rest of the primary nodes of A are labeled alternatingly, which is possible because cycle A has at least three primary nodes. Let the two primary nodes neighboring v_0 in cycle C be labeled 1 and 2. Then v_0 must be labeled 0 in order for each node in cycle A to have a neighbor with the same label. Then if cycle C is labeled alternatingly, we will violate (*).

If both cycles have only 2 primary nodes, it is possible for this design to have $k = 4$ and $m = 3$ and satisfy (*). See Fig. 14.

Case (3c): Endpoint of B is some node of A different from v_0 (see Fig. 13(c))

Two redundant nodes in the design have degree at least 3. Call them v_0 and v_1. Cycle A and path B make up three distinct paths which go from v_0 to v_1, which we will refer to as E, F and G. As long as no two paths have only 1 primary node, then we can find an alternating label and use Lemma 29.

Label $u_{i,X}$ to be the primary node neighboring v_i and in path X. If all paths E, F, and G have two or more primary nodes, assign labels $0, 1, 2$ to $u_{0,E}, u_{0,F}, u_{0,G}$ and $1, 2, 0$ to $u_{1,E}, u_{1,F}, u_{1,G}$. Each path can be labeled alternatingly.

If there is one path with only one primary node, say path E, assign labels $0, 1, 2$ to $u_{0,E}, u_{0,F}, u_{0,G}$ and $2, 1$ to $u_{1,F}, u_{1,G}$. Each path can be labeled alternatingly.

If two paths both have one primary node, say E and F, as long as the third path G has at least 3 primary nodes, the design can be labeled alternatingly. We can label the two primary nodes cycle created by paths E and F the values 0 and 1. Then since G has at least 3 primary nodes, we can label $u_{0,G}$ and $u_{1,G}$ the value 2 and label the rest of G alternatingly.

If G only has 2 primary nodes, then this is a design on $k = 4$ and $m = 3$ which satisfies (*). See Fig. 15.

Note that the two exceptions with $k = 4$ are precisely the minimal non-trivial 1 defect correcting designs. One of these designs was discussed in Section III-A. □

B. Proof of Proposition 21

Before we present the proof of Proposition 21, we will first show the following lemma:

Lemma 30: Fix $P_X \in \frac{1}{k}\mathbb{Z}$ *and* $s \in \mathbb{Z}_+$. *Let* $\underline{L} \sim \mathrm{Mult}(s, P_X)$, *cf.* (23), *and* $\underline{L}^{(k)} \sim \mathrm{HyperGeom}(k, s, P_X)$, *cf.* (27). *Then we have the following total variation estimate*[12]*:*

$$\mathsf{TV}(P_{\underline{L}^{(k)}}, P_{\underline{L}}) \le \frac{s^2}{2k}. \tag{57}$$

Similarly, if $\underline{M} \sim \mathrm{Mult}(s-1, P_X)$ *and* $\underline{M}^{(k-1)}$ *has distribution*

$$\mathbb{P}[\underline{M}^{(k-1)} = \underline{m}] = \frac{\binom{k\pi_1}{m_1} \cdots \binom{k\pi_j - 1}{m_j} \cdots \binom{k\pi_q}{m_q}}{\binom{k-1}{s-1}} \tag{58}$$

for an arbitrary j, *we also have*

$$\mathsf{TV}(P_{\underline{M}}, P_{\underline{M}^{(k-1)}}) \le \frac{s^2}{2k}. \tag{59}$$

Proof: Standard estimate for total variation via coupling states that for any joint distribution $P_{\underline{L},\underline{L}^{(k)}}$:

$$\mathsf{TV}(P_{\underline{L}^{(k)}}, P_{\underline{L}}) \le \mathbb{P}[\underline{L}^{(k)} \ne \underline{L}]. \tag{60}$$

Notice that $\underline{L}^{(k)}$ encodes the color distribution after sampling s balls from a collection of k colored balls (with composition given by P_X) without replacement, while $\underline{L}$ is the color distribution for sampling s balls with replacement. Let us couple these two samples as follows. Number all balls from 1 to k and define an infinite string of i.i.d. uniform $X_i \in [k]$. Let our sample with replacement be the balls with indices $X_1, \ldots, X_s$, while the sample without replacement be the balls $X_1, X_{i_2}, \ldots, X_{i_s}$ where i_t denotes the first element of the sequence where the t-th unique index appeared (e.g., for $X = (1, 2, 2, 3, \ldots)$ we have $i_2 = 2$, $i_3 = 4$, etc). Now the two samples are going to be different only if $X_1, \ldots, X_s$ are not distinct and this happens with probability at most

$$\sum_{i=1}^{s-1} \frac{i}{k} = \frac{s^2 - s}{2k} < \frac{s^2}{2k}. \tag{61}$$

This proves (57). For (59) modify distribution of X sequence by setting $X_1 = j$ and the rest are still i.i.d. uniform on $[k]$. Then $\underline{M}$ is the color composition of $X_2, \ldots, X_s$ while $\underline{M}^{(k-1)}$ is the color composition of $X_{i_2}, \ldots, X_{i_s}$. Again, $X_2, \ldots, X_s$ are not all distinct with probability at most (61). □

Proof of (28)*:* Simply by definition we have $F_{k,n}(P_S) \le F_k(P_S)$, so we focus on the opposite direction. First, we show that if $\underline{L}^{(k)} \sim \mathrm{HyperGeom}(s, k, [\pi_1, \cdots, \pi_q])$, for any function $f : \underline{L}^{(k)} \to \mathbb{R}$ and any fixed $j \in [q]$ we have

$$\frac{1}{\pi_j}\mathbb{E}[L_j^{(k)} f(\underline{L}^{(k)})] = \mathbb{E}[S \cdot f(\underline{M}^{(k-1)} + \underline{e}_j)], \tag{62}$$

where $\underline{e}_j$ is a vector with one in j-th position and the rest zeros, and $\underline{M}^{(k-1)}$ has hypergeometric distribution

$$\mathbb{P}[\underline{M}^{(k-1)} = \underline{m}] = \frac{\binom{k\pi_1}{m_1} \cdots \binom{k\pi_j - 1}{m_j} \cdots \binom{k\pi_q}{m_q}}{\binom{k-1}{s-1}}. \tag{63}$$

To that end, simply notice that

$$\frac{1}{\pi_j}\mathbb{E}[L_j^{(k)} f(\underline{L}^{(k)})]$$
$$= \sum_s P_S(s) \sum_{\underline{\ell}} \frac{\ell_j}{\pi_j} \frac{\binom{k\pi_1}{\ell_1} \cdots \binom{k\pi_q}{\ell_q}}{\binom{k}{s}} f(\underline{\ell}) \tag{64}$$
$$= \sum_s P_S(s) \sum_{\underline{m}} s \frac{\binom{k\pi_1}{m_1} \cdots \binom{k\pi_j - 1}{m_j} \cdots \binom{k\pi_q}{m_q}}{\binom{k-1}{s-1}} f(\underline{m} + \underline{e}_j). \tag{65}$$

Now fix $P_X \in \frac{1}{k}\mathbb{Z}$ and $P_{Y^*|\underline{L}^{(k)}}$ to be the optimal distributions achieving $F_k(P_S)$ in (25). By rounding there must exist $P_{Y_n|\underline{L}^{(k)}} \in \frac{1}{n}\mathbb{Z}$ so that $|P_{Y_n|\underline{L}^{(k)}}(j|\underline{\ell}) - P_{Y^*|\underline{L}^{(k)}}(j|\underline{\ell})| \le \frac{1}{n}$ for every $\underline{\ell}$. Then for any fixed $j \in [q]$ we have in view of (62)

$$\left|\frac{1}{P_X(j)}\mathbb{E}[L_j^{(k)} \mathbb{1}\{Y^* = j\}] - \frac{1}{P_X(j)}\mathbb{E}[L_j^{(k)} \mathbb{1}\{Y_n = j\}]\right| \le \frac{\mathbb{E}[S]}{n}. \tag{66}$$

Taking $\min_j$ of (66) recovers the lower bound in (28).

We proceed to proving (29). Fix P_S and let

$$h(P_X, P_{Y|\underline{L}}, j) \triangleq \frac{1}{P_X(j)}\mathbb{E}\left[L_j \mathbb{1}\{Y = j\}\right], \tag{67}$$

where given $S = s$ we have $\underline{L} \sim \mathrm{Mult}(s, P_X)$, cf. (22). Similar to (62) we have

$$h(P_X, P_{Y|\underline{L}}, j) = \mathbb{E}\left[S \cdot P_{Y|\underline{L}}(j|\underline{M} + \underline{e}_j)\right], \tag{68}$$

where this time given $S = s$ we have $\underline{M} \sim \mathrm{Mult}(s-1, P_X)$. Now, for $P_X \in \frac{1}{k}\mathbb{Z}$ define also

$$h_k(P_X, P_{Y|\underline{L}}, j) \triangleq \frac{1}{P_X(j)}\mathbb{E}\left[L_j^{(k)} \mathbb{1}\{Y = j\}\right], \tag{69}$$

where given $S = s$ we have $\underline{L}^{(k)} \sim \mathrm{HyperGeom}(s, k, P_X)$. From (62), (68) and Lemma 30 (namely (59)) we have then

$$|h(P_X, P_{Y|\underline{L}}, j) - h_k(P_X, P_{Y|\underline{L}}, j)| \le \frac{\mathbb{E}\left[S^3\right]}{2k}. \tag{70}$$

Finally, since

$$(P_X, P_{Y|\underline{L}}) \mapsto \min_j h(P_X, P_{Y|\underline{L}}, j) \tag{71}$$

is uniformly continuous on a compact set, we also have

$$(P_X, P_{Y|\underline{L}}) \mapsto \max_{P_{Y|\underline{L}}} \min_j h(P_X, P_{Y|\underline{L}}, j) \tag{72}$$

is uniformly continuous by Proposition 31. Hence for some $\epsilon_k \to 0$ we have

$$\left|\min_{P_X \in \frac{1}{k}\mathbb{Z}} \max_{P_{Y|\underline{L}}} \min_j h(P_X, P_{Y|\underline{L}}, j) - \min_{P_X} \max_{P_{Y|\underline{L}}} \min_j h(P_X, P_{Y|\underline{L}}, j)\right| \le \epsilon_k. \tag{73}$$

Using (70) to replace h with h_k in the first term of the latter we get (29). □

Proposition 31: Let $f : X \times Y \to \mathbb{R}$ *where* X *and* Y *are compact and* f *is uniformly continuous. Then* $\max_y f(x, y)$ *is uniformly continuous on* X.

[12] Total variation distance TV for probability measures P and Q on sigma algebra $\mathcal{F}$ defined as $\mathsf{TV}(P, Q) = \sup_{A \in \mathcal{F}} |P(A) - Q(A)|$.

Proof: Let $h(x) = \max_y f(x, y)$. Because f is uniformly continuous, for every $\epsilon > 0$ there exists a δ so that if the distance between (x_1, y_1) and (x_2, y_2) is less than δ, then $|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. We want to show that for h, the same δ can be used for each ϵ. Suppose there exists values $x, x' \in X$ where $|h(x') - h(x)| > \epsilon$ and $|x - x'| < \delta$. Assume that $h(x') > h(x)$. There exists a value of y so that $f(x', y) = h(x')$. Since $|f(x', y) - f(x, y)| \leq \epsilon$ then $h(x') = f(x', y) \leq f(x, y) + \epsilon \leq h(x) + \epsilon$, which is a contradiction. □

C. Upper Bound on $F_k(P_S)$

Lemma 32: For any $P_S \in \mathbb{Q}$ with finite support,

$$F_k(P_S) \leq F(P_S). \tag{74}$$

Proof: Fix $P_S \in \mathbb{Q}$ with finite support. First, we will show that $F_k\ (P_S) \leq F_{2k}(P_S)$. Using Proposition 22 and (28) from Proposition 21, for each k there exists a sequence of subset designs G_i which are $(k, m_i, t_i, E_i)_q$-designs with $E_i = m_i \mathbb{E}[S]$, and $\frac{t_i k}{m_i} \to F_k(P_S)$.

For each G_i, we will construct subset design G_i' on $2k$ primary nodes by copying (see Proposition 13) two copies of G_i. Then G_i' is a $(2k, 2m_i, t_i, 2E_i)_q$-design. By Proposition 23, for each G_i', there exists a subset design G_i'' which is a $(2k, 2m_i \cdot\ (2k)!, t_i \cdot\ (2k)!, 2E_i \cdot\ (2k)!)_q$*-design.*

$$F_{2k}(P_S) \geq \lim_{i\to\infty} \frac{t_i \cdot (2k)! 2k}{2m_i \cdot (2k)!} = \lim_{i\to\infty} \frac{t_i k}{m_i} = F_k(P_S). \tag{75}$$

Then, $F_k\ (P_S) \leq F_{2k}(P_S) \leq F_{4k}(P_S) \leq F_{8k}(P_S) \leq F_{16k}(P_S) \cdots$. Since $F_{2^i k}(P_S) \to F(P_S)$ monotonically with convergence given by (29) from Proposition 21, this gives the desired result. □

D. Proof of Proposition 24

Proof: Fixed P_S with finite support and let $c = \mathbb{E}[S]$. Let $\hat{P}_S(s) = \frac{P_S(s)s}{\sum_s P_S(s)s} = \frac{1}{c} P_S(s)s$. We can substitute in $\hat{P}(s)$ and take the expectation with respect to $\hat{P}_S$ instead of P_S by adjusting (21) to

$$F(\hat{P}_S) = \min_{P_X} \max_{P_{Y|L}} \min \left\{ \frac{c}{P_X(0)} \frac{L_0}{S} \mathbb{1}\{Y = 0\}, \right.$$
$$\left. \frac{c}{P_X(1)} \frac{L_1}{S} \mathbb{1}\{Y = 1\} \right\}. \tag{76}$$

Let the redundant node ratio of a redundant node with type (ℓ_0, ℓ_1) be $\nu = \frac{\ell_0}{\ell_0 + \ell_1}$. Suppose that $P_{Y|L}$ is a labeling so that

1) $P_{Y|L}(1|\underline{\ell}_a) > 0$ where $\underline{\ell}_a$ so that $\ell_0 + \ell_1 = s_a$ and has ratio ν_a
2) $P_{Y|L}(0|\underline{\ell}_b) > 0$ where $\underline{\ell}_b$ so that $\ell_0 + \ell_1 = s_b$ and has ratio ν_b
3) $\nu_a > \nu_b$

Let $P_{Y'|L}$ be equivalent to $P_{Y|L}$ except that

$$P_{Y'|L}(1|\underline{\ell}_a) = P_{Y|L}(1|\underline{\ell}_a) - \alpha \hat{P}_S(s_b) P_{\underline{L}|S}(\underline{\ell}_b|s_b) \tag{77a}$$
$$P_{Y'|L}(0|\underline{\ell}_a) = P_{Y|L}(0|\underline{\ell}_a) + \alpha \hat{P}_S(s_b) P_{\underline{L}|S}(\underline{\ell}_b|s_b) \tag{77b}$$
$$P_{Y'|L}(1|\underline{\ell}_b) = P_{Y|L}(1|\underline{\ell}_b) + \alpha \hat{P}_S(s_a) P_{\underline{L}|S}(\underline{\ell}_a|s_a) \tag{77c}$$
$$P_{Y'|L}(0|\underline{\ell}_b) = P_{Y|L}(0|\underline{\ell}_b) - \alpha \hat{P}_S(s_a) P_{\underline{L}|S}(\underline{\ell}_a|s_a) \tag{77d}$$

for an appropriate $\alpha > 0$ which is small enough so that $P_{Y'|L}$ is still a valid distribution. Compared to $P_{Y|L}$, $P_{Y'|L}$ increases both quantities in the brackets in (76). So $P_{Y|L}$ cannot be optimal and any optimal $P_{Y|L}$ must have the form of (42).

For two redundant node type ratios where $\nu_a = \nu_b$, we can also see from (77a)-(77d) that there is a value of α (possibly negative unlike above) so that $P_{Y'|L}(0|\underline{\ell}_a) = P_{Y'|L}(0|\underline{\ell}_b)$ and the value of (76) is not affected by the change. □

E. Numerical Results Derivation

Here we develop upper and lower bounds for the expression found in Theorem 6 (the particular case when $\mathcal{X} = \{0, 1\}$).

1) Almost Tight Lower Bound: Our lower bound for the boundary of $\mathcal{R}_\infty$ will be parametrized by c. To get this lower bound, we want to find an upper bound for

$$Z^*(c) = \max_{P_S : \mathbb{E}[S] = c} F(P_S). \tag{78}$$

For notation, let $\lambda = P_X(0)$ and $1 - \lambda = P_X(1)$. Let $\underline{\ell} = (\ell_0, \ell_1)$ and $f(\ell_0, \ell_1) = f(\underline{\ell}) = P_{Y|L}(0|\underline{\ell})$ where f can take any value between $[0, 1]$. For a fixed λ and s, define random variable $\underline{M}(s, \lambda) = (V, s - 1 - V)$ where $V \sim \text{Bino}(s - 1, \lambda)$ and $\underline{e}_0 = (1, 0)$ and $\underline{e}_1 = (0, 1)$ according to Lemma 30 and the proof of Proposition 21. (Just for clarity in this section, we added arguments in paranthesis for $\underline{M}$.) Fix P_S to have finite support.

First, we have that

$$F(P_S) = \min_{0\leq\lambda\leq1} \max_{0\leq f\leq1} \min \left\{ \frac{1}{\lambda} \mathbb{E}[L_0\ f(\underline{L})], \right.$$
$$\left. \frac{1}{1 - \lambda} \mathbb{E}[L_1(1 - f(\underline{L}))] \right\} \tag{79}$$
$$= \min_{0\leq\lambda\leq1} \max_{0\leq f\leq1} \min_{0\leq\alpha\leq1} \alpha \mathbb{E}[S \cdot f(\underline{M}(S, \lambda) + \underline{e}_0)]$$
$$+(1 - \alpha)\mathbb{E}[S \cdot\ (1 - f(\underline{M}(S, \lambda) + \underline{e}_1))] \tag{80}$$
$$\leq \min_{0\leq\lambda\leq1} \frac{1}{2} \mathbb{E}\left[S \cdot \max_{0\leq f\leq1} (1\ + f(\underline{M}(S, \lambda) + \underline{e}_0) \right.$$
$$\left. - f(\underline{M}(S, \lambda) + \underline{e}_1)) \right] \tag{81}$$
$$= \min_{0\leq\lambda\leq1} \frac{1}{2} \mathbb{E}\left[S \cdot (1 + \max_{0\leq\ell_0\leq S-1} \mathbb{P}[\underline{M}(S, \lambda) \right.$$
$$\left. = (\ell_0, S - 1 - \ell_0)]) \right] \tag{82}$$
$$\overset{\triangle}{=} \min_{0\leq\lambda\leq1} \mathbb{E}[\phi(S, \lambda)] \tag{83}$$

where in (80) we use (68) and convexify the minimum using α, and then (81) follows by setting $\alpha = \frac{1}{2}$. To get (82), notice that for a fixed s

$$\mathbb{E}[1 + f(\underline{M}(s, \lambda) + \underline{e}_0) - f(\underline{M}(s, \lambda) + \underline{e}_1)] \tag{84}$$
$$= 1 + \sum_{\ell_0=0}^{s-1} \mathbb{P}[\underline{M}(s, \lambda) = (\ell_0, s-1-\ell_0)] f(\ell_0+1, s - 1 - \ell_0)$$
$$- \sum_{\ell_0=0}^{s-1} \mathbb{P}[\underline{M}(s, \lambda) = (\ell_0, s - 1 - \ell_0)] f(\ell_0, s - \ell_0) \tag{85}$$
$$= 1 + \mathbb{P}[\underline{M}(s, \lambda) = (x, s - 1 - x)] f(x + 1, s - 1 - x)$$
$$+\mathbb{P}[\underline{M}(s, \lambda) = (x + 1, s - x)](1 - f(x + 1, s - 1 - x)). \tag{86}$$

By Proposition 24, the optimal f must have a threshold solution. We can express this threshold solution by letting x be be smallest value of ℓ_0 where $f(\ell_0, s-\ell_0)$ is non-zero. Applying the cancellations to (85), we get that only two terms remains. The value of f which obtains the maximum must be where only the maximum value of $\mathbb{P}[\underline{M}(s,\lambda) = (x, s-1-x)]$ over all x appears in (86), and this gives (82).

We will bound

$$\begin{aligned} Z^*(c) &\le \max_{P_S:\mathbb{E}[S]=c} \min_{0\le\lambda\le 1} \mathbb{E}[\phi(S,\lambda)] \qquad (87)\\ &\le \max_{P_S:\mathbb{E}[S]=c} \min_{\lambda\in L_n} \mathbb{E}[\phi(S,\lambda)] \qquad (88)\\ &\stackrel{\triangle}{=} Z'_n(c) \qquad (89)\end{aligned}$$

where we defined[13]

$$L_n = \left\{ \frac{\lfloor s/2 \rfloor}{s} : \text{ where } 1 < s \le 2n \right\}.$$

Note that increasing n makes the approximation tighter. Index the elements of L_n as λ_i where $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{3}, \lambda_3 = \frac{2}{5}, \ldots, \lambda_n = \frac{n-1}{2n-1}$, so that $\min_{\lambda\in L_n} \mathbb{E}[\phi(S,\lambda)] = \min_i \mathbb{E}[\phi(S,\lambda_i)]$.

The quantity $Z'_n(c)$ is equivalent to maximizing the value of t under the constraints that $\mathbb{E}[\phi(S,\lambda_i)] \ge t$ for all $1 \le i \le n$ and $\mathbb{E}[S] = c$. We can substitute

$$\begin{aligned}\phi(s,\lambda_i) &= \frac{s}{2}\left(1 + \max_{1\le\ell_0\le s} \mathbb{P}[\underline{M}(s,\lambda_i) = (\ell_0, s-1-\ell_0)]\right) \qquad (90)\\ &\stackrel{\triangle}{=} \frac{s}{2}\left(1+\psi(s,\lambda_i)\right). \qquad (91)\end{aligned}$$

Then $\mathbb{E}[\phi(S,\lambda_i)] = \frac{1}{2}\mathbb{E}[S] + \frac{1}{2}\mathbb{E}[S\cdot\psi(S,\lambda_i)] = \frac{c}{2} + \frac{1}{2}\mathbb{E}[S\cdot\psi(S,\lambda_i)]$. Note that $\psi(S,\lambda_i) \to 0$ as $s\to\infty$ for all i.

For any value of $\pi_i \ge 0$, where $1 \le i \le n$, $\eta \ge 0$ and $\mu \ge 0$, we can define

$$\begin{aligned} &Z''_n(c,\pi_1,\ldots,\pi_n,\eta,\mu)]\\ &\stackrel{\triangle}{=} \max_{P_s(s)\ge 0,\forall s} t + \sum_i \pi_i\left(\frac{c}{2} + \sum_{s=1}^{\infty} P_S(s)\frac{s}{2}\psi(s,\lambda_i) - t\right)\\ &\qquad -\eta\left(\sum_{s=1}^{\infty} P_S(s)s - c\right) - \mu\left(\sum_{s=1}^{\infty} P_S(s) - 1\right). \qquad (92)\end{aligned}$$

Consider the set of π_i, η, μ which is the solution to the dual problem

$$\begin{aligned} &\textbf{minimize } \frac{c}{2} + \eta c + \mu \qquad (93a)\\ &\textbf{subject to } \sum_i^n \frac{1}{2}\pi_i\psi(s,\lambda_i) - \eta - \mu\frac{1}{s} \le 0, s\in\mathbb{Z}_+ \qquad (93b)\\ &\qquad \sum_{i=1}^n \pi_i - 1 = 0 \qquad (93c)\\ &\qquad \eta \ge 0, \mu\ge 0, \pi_i \ge 0, 1\le i\le n. \qquad (93d)\end{aligned}$$

Such an optimization has a solution which is easy to find despite having infinitely many constraints. The constraints (93b) will hold for all s greater than some s_0 because $\psi(s,\lambda_i)\to 0$. By choosing a large enough s_0, we can solve the optimization by replacing it with an optimization where only the first s_0 constraints in (93b) are present.[14] Set the values of π_i, η, μ in (92) to be the values which obtain the minimum for (93a)-(93d). Select a value of $s_1 \in \mathbb{Z}_+$. Then

$$\begin{aligned} Z'_n(c) &\le Z''_n(c,\pi_1,\ldots,\pi_n,\eta,\mu) \qquad (94)\\ &= \max_{P_s(s)\ge 0,\forall s} t\\ &\quad + \sum_i \pi_i\left(\frac{c}{2} + \sum_{s=1}^{\infty} P_S(s)\frac{s}{2}\psi(s,\lambda_i) - t\right)\\ &\quad -\eta\left(\sum_{s=1}^{\infty} P_S(s)s - c\right) - \mu\left(\sum_{s=1}^{\infty} P_S(s) - 1\right) \qquad (95)\\ &= \max_{P_s(s)\ge 0,\forall s} \sum_i \pi_i\left(\sum_{s=s_1}^{\infty} P_S(s)\frac{s}{2}\psi(s,\lambda_i)\right)\\ &\quad -\eta\left(\sum_{s=s_1}^{\infty} P_S(s)s\right) - \mu\left(\sum_{s=s_1}^{\infty} P_S(s)\right)\\ &\quad + \frac{c}{2} + \eta c + \mu \qquad (96)\\ &\le \max_{s>s_1}\max_i \frac{c}{2}\psi(s,\lambda_i) + \frac{c}{2} + \eta c + \mu. \qquad (97)\end{aligned}$$

Since $\psi(s,\lambda_i) \to 0$, the optimal $\frac{c}{2} + \eta c + \mu$ given by (93a)-(93d) is an upper bound to $Z'_n(c)$ and hence also to $Z^*(c)$. This computes a lower bound on $\mathcal{R}_\infty$. In Fig. 10, we found the lower bound using $n = 10$.

2) Upper Bounds: To show a point in $\mathcal{R}_\infty$ is achievable, it is sufficient to find a set of masses P_S that achieves that point. Searching all possible masses P_S is not computationally efficient. It turns out we can get decently close to the lower bound approximation by using the same masses which are solutions to $Z'_n(c)$ for each c when restricting P_S to only have finite support. While these results are close to the almost tight converse bound, they are not necessarily the best known. A few best known achievable points were found by simple search. The results are plotted in Fig. 10 and shown in Table I.

F. Proof of Corollary 27

Proof: The Hamming block achieves the point $(\frac{3}{2}, \frac{2}{3})$ in $\mathcal{R}^3_\infty$. The proof that this is a corner point amounts to computing the region $\mathcal{R}^3_\infty$.

To solve for $\mathcal{R}^3_\infty$, we will first simplify the expression for $F_3(P_S)$. For any P_S on $s \in [3]$, the labeling of primary nodes which gives the minimum value of $F_3(P_S)$ is when $P_X(0) = \frac{2}{3}$ and $P_X(1) = \frac{1}{3}$ (or these flipped). With this insight, we can simplify $F_3(P_S)$ to solve for the optimal P_S

[13] L_n is defined so that $L_n = \{\lambda \in (0, \frac{1}{2}] : \phi(s,\lambda) \le \phi(s,\lambda')$ for some $1 < s \le 2n$ and $\forall\lambda' \in [0,1]\}$ which is the set of all λ which minimizes $\phi(s,\lambda)$ for some $1 < s \le 2n$.

[14] We can show that for each c, only considering constraints (93b) for $s \le 16c$ is more than sufficient. All other infinite constraints can be removed without affecting the optimal solution.

given any parameter $\mathbb{E}[S] = c$ for some $1 \leq c \leq 3$. Let $P_{Y|\underline{L}^{(3)}}(j|\ell_0, \ell_1)$ denote the proportion of redundant nodes of type $\underline{\ell} = (\ell_0, \ell_1)$ to label j.

$$F_3(P_S) = \max_{P_{Y|\underline{L}^{(3)}}} \min_{j \in \{0,1\}} \frac{1}{P_X(j)} \mathbb{E}\left[L_j \mathbb{1}\{Y = j\}\right] \tag{98}$$

$$= \max_{P_{Y|\underline{L}^{(3)}}} \min_{j \in \{0,1\}} \left\{ \frac{1}{P_X(j)} \sum_{s=1}^{3} P_S(s) \sum_{\underline{\ell}} \ell_j P_{\underline{L}^{(3)}|S,P_X}(\underline{\ell}|s, P_X) P_{Y|\underline{L}^{(3)}}(j|\underline{\ell}) \right\} \tag{99}$$

$$\begin{aligned} = \max_{P_{Y|\underline{L}^{(3)}}} \min \Bigg\{ & \frac{3}{2} \Bigg(P_S(1) \frac{2}{3} P_{Y|\underline{L}^{(3)}}(0|1,0) \\ & + P_S(2) \left[\frac{2}{3} P_{Y|\underline{L}^{(3)}}(0|1,1) + 2\frac{1}{3} P_{Y|\underline{L}^{(3)}}(0|2,0) \right] \\ & + P_S(3) 2 P_{Y|\underline{L}^{(3)}}(0|2,1) \Bigg), \\ & \frac{3}{1} \Bigg(P_S(1) \frac{1}{3} P_{Y|\underline{L}^{(3)}}(1|0,1) \\ & + P_S(2) \frac{2}{3} P_{Y|\underline{L}^{(3)}}(1|1,1) \\ & + P_S(3) P_{Y|\underline{L}^{(3)}}(1|2,1) \Bigg) \Bigg\} \end{aligned} \tag{100}$$

To get (100), we expanded the summation into each term replacing ℓ_j and $P_{\underline{L}^{(3)}|S,P_X}(\underline{\ell}|s, P_X)$ with their numerical values. Since $P_X(0) = \frac{2}{3}$ and $P_X(1) = \frac{1}{3}$ and $k = 3$, we only need the $P_{Y|\underline{L}^{(3)}}$ terms for which $\underline{\ell}$ is a subset on 2 zeros and 1 one.

We will first solve for the portion of $\mathcal{R}_\infty^3$ where $\varepsilon > \frac{3}{2}$.

Set $\mathbb{E}[S] = 3$. There is a unique point of the form $(\frac{3}{\eta}, \frac{1}{\eta})$ for some $\eta > 0$ which is a boundary point of the convex region $\mathcal{R}_\infty^3$. The only distribution P_S which can achieve $\mathbb{E}[S] = 3$ is when $P_S(3) = 1$ and $P_S(s) = 0$ for all other $s \neq 3$. With this P_S, we get that $F_3(P_S) = \frac{3}{2}$. Since no other P_S is possible, the point

$$\left(\frac{\mathbb{E}[S]}{F_3(P_S)}, \frac{1}{F_3(P_S)} \right) = \left(2, \frac{2}{3} \right) \tag{101}$$

must be the boundary point of the form $(\frac{3}{\eta}, \frac{1}{\eta})$ in $\mathcal{R}_\infty^3$. The line of points between this value and the value given by the Hamming block is achievable by convexity and by Claim 2 of Proposition 4 they must be optimal.

For the remaining portion of the region, we want to fixed a $1 < c < \frac{9}{4}$ (the Hamming block has $\mathbb{E}[S] = \frac{9}{4}$), and solve for $P_S^* = \operatorname{argmax}_{P_S : \mathbb{E}[S] = c} F_3(P_S)$ and determine $F_3(P_S^*)$.

Note that it is optimal to set $P_{Y|\underline{L}^{(3)}}(0|\ell_0, 0) = 1$ and $P_{Y|\underline{L}^{(3)}}(1|0, \ell_1) = 1$. Then we can simplify notation by letting $P_{Y|\underline{L}^{(3)}}(0|1,1) = x_{1,1}$ and $P_{Y|\underline{L}^{(3)}}(0|2,1) = x_{2,1}$. We can simplify (100) by applying the constraints that $\sum_{s=1}^{3} P_S(s) = 1$ and $\sum_{s=1}^{3} P(s)s = c$. At the maximum point, the two quantities after the minimum must be equal. Simplifying the equation with these constraints, we have

$$\begin{aligned} F_3(P_S^*) = \max_{x_{1,1}, x_{2,1} \in [0,1]} & \frac{c - 1 - (3c-3)x_{1,1}}{6x_{2,1} - 6x_{1,1} - 1} (3x_{2,1} - 2x_{1,1} - 1) \\ & + (c-1)x_{1,1} + 1 \end{aligned} \tag{102}$$

under the constraints that the variables are in [0, 1].

The optimal labeling must have that either $x_{1,1} = 0$ and $x_{2,1} \in [0,1]$ or that $x_{1,1} \in [0,1]$ and $x_{2,1} = 1$ by Proposition 24. We try the cases $x_{1,1} = 0$ and $x_{2,1} = 1$ and take derivatives to solve for the best value of $x_{2,1}$ or $x_{1,1}$. For any value of c we pick,[15] the point $(\frac{c}{F_3(P_S^*)}, \frac{1}{F_3(P_S^*)})$ lies on the line between the point achievable by the repetition design and the Hamming block. By convexity, it must be that all points on the line between the values achievable by the repetition design and the Hamming block are optimal. □

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