On the Excess Distortion Exponent of the Quadratic-Gaussian Wyner-Ziv Problem

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Abstract—An achievable excess distortion exponent for compression of a white Gaussian source by dithered lattice quantization is derived. We show that for a required distortion level close enough to the rate-distortion function, and in the high-rate limit, the exponent equals the optimal quadratic-Gaussian excess distortion exponent. Using this approach, no further loss is incurred by the presence of any source interference known at the decoder ("Wyner-Ziv side-information"). The derivation of this achievable exponent involves finding the exponent of the probability that a combination of a spherically-bounded vector and a Gaussian vector leaves the Voronoi cell of a good lattice.

I. INTRODUCTION

The excess distortion exponent specifies the exponential decay rate of the probability that the distortion exceed a prescribed threshold, as a function of the block length. It was defined and evaluated for discrete memoryless sources by Marton [7]. The extension of the exponent to quadratic-Gaussian (QG) case was carried out much later by Ihara and Kubo [3].

It is well known [10] that there is no rate loss in the QG Wyner-Ziv (WZ) problem, i.e., if part of the source is given as side-information (SI) at the decoder, and the unknown part is Gaussian, then the rate needed for conveying the source with some mean-squared error (MSE) is equal to the rate required for compressing the unknown source part with the same MSE (or, equivalently, the rate needed if the encoder also had access to the SI). However, the excess distortion exponent for this problem is as yet unknown, and in particular it is not clear whether there is a loss with respect to the excess-distortion exponent of the unknown part of the source. Recently, Kelly et al. [4] derived an achievable exponent for the problem, which indeed reflects a loss.

The rate-distortion function (RDF) can be achieved via a forward-channel materialization, where a dithered quantizer is seen as an additive noise component (see [13] and references therein). This forward-channel structure was used by Zamir et al. [14], in conjunction with a modulo-coarse-lattice operation, to present a solution to the QG WZ problem. This scheme has a practical advantage even in the no-interference case: if a nested-lattice structure is used for the quantization code and coarse lattice, then lattice quantization may be performed, i.e., the nearest-codeword search is done with respect to a periodic structure, yielding reduced complexity.

This encoding structure is presented in [14] as a dual to a capacity-achieving nested-lattice solution for the additive white Gaussian noise (AWGN) channel and dirty-paper problems. Later ([6], see also [9]) it was shown that by judiciously changing some multiplicative factors appearing in the scheme, the lattice approach also achieves (for high enough rate) the AWGN channel error exponent. It is tempting to believe that a similar variation on the nested-lattice source scheme will yield the QG excess distortion exponent, thus also proving that there is no loss in the exponent of the corresponding WZ problem. Unfortunately, as we show in this paper by straightforward optimization, such a variation of the nested-lattice source scheme does not achieve the QG excess-distortion exponent.

Our analysis provides an achievable excess distortion exponent using lattice quantization. The same exponent is achievable in the QG WZ problem with respect to the unknown source part, for an arbitrary known source part. Since in the high-rate limit, when the required distortion is close enough to the distortion-rate limit, this exponent reduces to the optimal exponent of [3] - this proves that at least in this limit there is no loss in the exponent of the QG WZ problem.

The derivation of the achievable exponent hinges on

finding the exponents of the probability that a combination of a vector bounded inside a sphere and an i.i.d. Gaussian vector leave either a sphere or the Voronoi cell of a good lattice. We find these exponents, based on the AWGN channel exponent analysis in [6], [9].

The rest of the paper is organized as follows. In Section II we formally state the problem and present the main result. In Section III we prove the result using a nested-lattice scheme, in terms of the spherical-Gaussian combination error exponent derived in Section IV. Finally, in Section V we discuss the gap between the lattice exponent and the QG excess distortion exponent.

II. PROBLEM STATEMENT AND MAIN RESULT

We consider compression of an n-dimensional Gausian vector $\mathbf{X}$, where the samples are i.i.d. Gaussian with zero mean and variance σ_X^2. The decoder produces a reconstruction $\hat{\mathbf{X}}$ based on an index from a code of rate R. When considering the QG WZ problem, the encoder does not have access to $\mathbf{X}$ directly, but it rather observes $\mathbf{X} + \mathbf{I}$, where $\mathbf{I}$ is an arbitrary signal ("interference"), available as non-causal SI at the decoder. [1]

The reconstruction quality is measured by the signal-to-distortion ratio (SDR) with respect to the mean-squared error, i.e.

$$\text{SDR}(\mathbf{X}, \hat{\mathbf{X}}) = \frac{n\sigma_X^2}{\|\hat{\mathbf{X}} - \mathbf{X}\|^2} \quad . \tag{1}$$

The average SDR, $\overline{\text{SDR}}$, is given by the same, replacing the norm in the denominator by its expectation; using this, the RDF is given by:

$$R(\overline{\text{SDR}}) = \frac{1}{2}\log(\overline{\text{SDR}}) \quad , \tag{2}$$

where here and elsewhere in this work the natural base is used for logarithms and exponents.

For any required SDR S and code $\mathcal{C}_n$ of block length n and rate $R > R(S)$, define the error probability as:

$$p_e(\mathcal{C}_n, S) = \Pr\{\text{SDR}(\mathbf{X}, \hat{\mathbf{X}}) < S\} \quad . \tag{3}$$

The excess distortion exponent is defined with respect to a sequence of codes indexed by the block length n:

$$E(R, S) = \sup_{\{\mathcal{C}_n\}} \lim_{n\to\infty} -\frac{1}{n}\log p_e(\mathcal{C}_n, S) \quad . \tag{4}$$

It is more convenient to express the exponent using an alternative pair of parameters: β, which is the SDR at

[1]A more common definition sees the sum $\mathbf{X} + \mathbf{I}$ as the source, and the SI as some vector jointly-Gaussian with the source. The interference point of view sees only the estimation error from the SI as the source. We choose to follow it, since it allows to to directly evaluate the loss in quantizing $\mathbf{X}$ due to the presence of $\mathbf{I}$.

the RDF, and the distortion factor μ, measuring the gap from the RDF (much like SNR_{norm} in channel coding):

$$\beta = \exp(2R)$$
$$\mu = \frac{\beta}{S} > 1 \quad . \tag{5}$$

Without interference, the result of [3] states that: [2]

$$E(\mu, \beta) = F(\mu) \triangleq \frac{\mu - \log(\mu) - 1}{2} \quad . \tag{6}$$

Note that this is the exponent of the probability that a Gaussian i.i.d. vector leaves a sphere, when the ratio between the second moment of the sphere and the variance of the vector (per dimension) is μ.

We now define two spherical-Gaussian exponents. Let $\boldsymbol{U}$ be on the surface of a sphere of radius n/μ_U, and let $\mathbf{N}$ be n-dimensional i.i.d. zero-mean Gaussian with variance $1/\mu_G$. Furthermore, let $\mathcal{B}$ and $\mathcal{V}$ be an n-dimensional sphere and and Voronoi region, respectively, both of volume 1. Then:

$$F(\mu_G, \mu_U) = -\lim_{n\to\infty} \frac{1}{n}\log \Pr\{\mathbf{N} + \boldsymbol{U} \notin \mathcal{B}\}$$
$$F'(\mu_G, \mu_U) = -\lim_{n\to\infty} \frac{1}{n}\log \Pr\{\mathbf{N} + \boldsymbol{U} \notin \mathcal{V}\} \quad . \tag{7}$$

We now state our main result in terms of these exponents.

Theorem 1: The exponent $\underline{E}(\mu, \beta)$ given by

$$\max_{\alpha_1, \alpha_2} \min\left[F'\left(\frac{\mu}{\alpha_1^2}, \beta\right), F\left(\frac{\mu}{(1 - \alpha_1\alpha_2)^2\beta}, \frac{1}{\alpha_2^2}\right)\right] \tag{8}$$

is achievable with any WZ interference $\mathbf{I}$. Furthermore, it is achievable using lattice quantization.

In Section IV we lower-bound the exponents F and F', for an appropriate choice of lattice sequences. For now we give a simple bound which is less tight, but still suffices for deriving the high-rate performance. By choosing $\alpha_1 = \alpha_2 = 1$ in (8), we have that

$$\underline{E}(\mu, \beta) \geq F'(\mu, \beta) \quad . \tag{9}$$

Now the effect of a combination noise can not be worse (in the exponential sense) than that of a Gaussian i.i.d. vector with the same total variance [2], i.e.,

$$F'(\mu_G, \mu_U) \geq E_p\left(\frac{\mu_G\mu_U}{\mu_G + \mu_U}\right) \tag{10}$$

where the Poltyrev exponent E_p [8] is given by:

$$E_p(\mu) = \begin{cases} F(\mu), & \mu \leq 2 \\ \frac{1}{2}\left(1 + \log\frac{\mu}{4}\right), & 2 < \mu \leq 4 \\ \frac{\mu}{8}, & 4 < \mu \ , \end{cases} \tag{11}$$

[2]Interestingly, this expression is not dual to channel coding, in the sense that there is no critical rate.
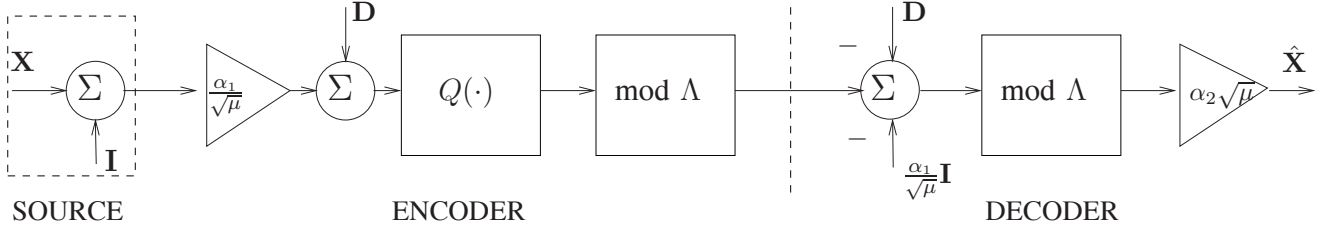
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Fig. 1: Nested-lattice coding scheme.

where F is given by (6). Combining (9),(10) and taking the limit of high β, we have the following.

Corollary 1: For any μ,

$$\lim_{\beta \to \infty} \underline{E}(\mu, \beta) \geq E_p(\mu) \quad .$$

Thus, in this limit and when $\mu \leq 2$, the achievable exponent for any interference $\mathbf{I}$ and using lattice quantization, approaches the QG excess distortion exponent $F(\mu)$.

III. PROOF USING A NESTED-LATTICE SCHEME

In this section we provide a constructive proof of Theorem 1, using the scheme depicted in Figure 1. We use an n-dimensional nested lattice structure $\Lambda_Q \supset \Lambda$ with nesting ratio R. Quantization is carried out with respect to the fine lattice Λ_Q and the result is then reduced modulo the coarse lattice Λ. Let the basic cells of the two lattices be $\mathcal{V}_Q$ and $\mathcal{V}$, respectively. The dither $\mathbf{D}$, known at both the encoder and decoder, is uniformly distributed over $\mathcal{V}_Q$. If one substitutes $\mu = 1$ and $\alpha_1 = \alpha_2 = (\beta - 1)/\beta$ (5), the scheme reduces to that of [14, Fig. 11], which achieves the RDF in the QG WZ problem (with a zero exponent).

By construction, the nesting ratio R is indeed the rate needed to describe the quantization point. We normalize the second moment of Λ to 1; it follows that the second moment of Λ_Q is $1/\beta$. By the properties of dithered quantization (see e.g. [12]),

$$Q(\mathbf{Y} + \mathbf{D}) - \mathbf{D} = \mathbf{Y} + \mathbf{Z} \quad , \tag{12}$$

where $\mathbf{Z}$ is independent of the input $\mathbf{Y}$ and uniform over the mirror image of $\mathcal{V}_Q$.

Define the combination vectors:

$$\mathbf{C}_1 = \frac{\alpha_1}{\sqrt{\mu}} \mathbf{X} + \mathbf{Z}$$
$$\mathbf{C}_2 = -(1 - \alpha_1 \alpha_2) \mathbf{X} + \alpha_2 \sqrt{\mu} \mathbf{Z} \quad . \tag{13}$$

In term of these, let the error events be:

$$E_1 = \{\mathbf{C}_1 \notin \mathcal{V}\}$$
$$E_2 = \left\{ \|\mathbf{C}_2\|^2 > \frac{\mu}{\beta} \right\} \quad . \tag{14}$$

Following [14], if E_1 did not occur then

$$\hat{\mathbf{X}} = \mathbf{X} + \mathbf{C}_2 \quad .$$

Consequently we can bound the excess distortion probability as follows:

$$\Pr\left\{ \|\hat{\mathbf{X}} - \mathbf{X}\|^2 > \frac{1}{\mu\beta} \right\} \leq \Pr\{E_1\} + \Pr\{E_2 | \overline{E_1}\}$$
$$\leq \Pr\{E_1\} + \Pr\{E_2\} \quad , \tag{15}$$

where the last inequality stems from the fact that the probability that $\mathbf{C}_2$ lies inside a sphere can only increase given that $\mathbf{C}_1$ lies within the convex region $\mathcal{V}$. Recalling the definition of the exponents (7) we observe that these two probabilities have exponential decay according to the two terms in $\underline{E}$ (8), if only the basic cell $\mathcal{V}_Q$ were bounded in a sphere of radius n/β, i.e. a sphere of the same second moment as the cell. This is asymptotically true for Rogers-good lattices (see e.g. [2]), and it can be shown that the exponential decay of the probabilities is just as if this was true for any n. For completing the proof, we are now only left with the task of specifying the exponents $F(\cdot, \cdot)$ and $F'(\cdot, \cdot)$, for an appropriate choice of lattices.

IV. EXPONENTS FOR COMBINATION VECTORS

In this section we address the exponents (7) for the combination of a spherical-bounded vector and a Gaussian vector, to leave either a sphere or a Voronoi cell. We start by presenting the spherical-region exponent.

Lemma 1: For any $1/\mu_G + 1/\mu_U < 1$, the exponent F (7) is given by:

$$2F(\mu_G, \mu_U) = \frac{\mu_G}{\mu_U}(\mu_U - 2h\sqrt{\mu_U} + 1) - \log(h\sqrt{\mu_U}) - 1 \quad , \tag{16}$$

where

$$h \triangleq \frac{\sqrt{\mu_U + 4\mu_G^2} - \sqrt{\mu_U}}{2\mu_G} \quad .$$

The proof, not included in this version, is based upon decomposing the Gaussian noise $\mathbf{N}$ into a component aligned with the direction of the bounded vector and an orthogonal component; see e.g. [1], [9, Sec. III-A].
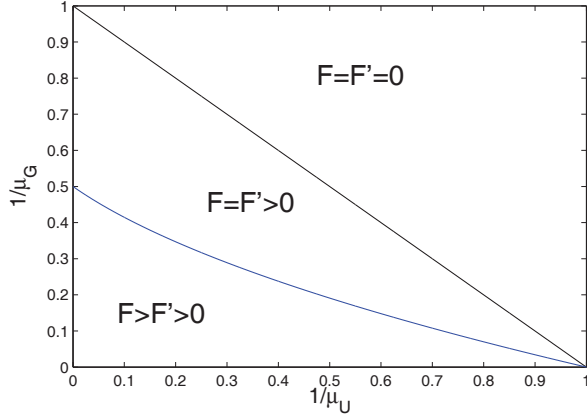
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Fig. 2: Regions of the combination-vector exponents.

For the Voronoi-region exponent F', we express our result also in terms of the straight-line exponent of the AWGN channel:

$$E_{sl}(R,S) = R_C - R + 1 - \exp(2R_C)$$
$$+ \frac{1}{2} \log\left(\exp(2R_C) - \frac{S}{2}\right) + \frac{S}{2} \quad , \quad (17)$$

where R is the rate, S is the signal-to-noise ratio and

$$R_C = \log\left(1 + \frac{S}{2} + \sqrt{1 + \frac{S^2}{4}}\right) \quad . \quad (18)$$

Lemma 2: There exists a sequence of lattices Λ, with Voronoi cells $\mathcal{V}$, such that for any $1/\mu_G + 1/\mu_U < 1$ the exponent F' (7) associated with $\mathcal{V}$ satisfies: $F'(\mu_G, \mu_U) \geq E_{sl}(R,S)$ for

$$S = \frac{(\gamma+1)^2 - (\gamma-1)\sqrt{\gamma^2 + 6\gamma + 1}}{2\gamma} \quad (19)$$

(where $\gamma = \mu_G/\mu_U$), and

$$R = -\log\left[\sqrt{\mu_U}\left(\sqrt{1 + \frac{S^2}{4}} - \frac{S}{2}\right)\right] \quad . \quad (20)$$

If $R > R_C$ (18) then $F'(\mu_G, \mu_U) = F(\mu_G, \mu_U)$ (16).

This lemma can be proven using a geometrical derivation, similar to what was done in [6], [9] for the AWGN exponents; in order to avoid repeating the rather complicated exposition, we use the following equivalence, following directly from the exposition in [6].

Proposition 1: An exponent $E_\Lambda(R,S)$ for the modulo-lattice additive noise channel is achievable by the scheme of [6, Sec. 3.1] if and only if there exist some $0 < \alpha < 1$, $0 < R < C$ and $S > 0$ such that $F'(\mu_U, \mu_G)$ is achievable for:

$$\frac{1}{\nu_G} = \alpha^2 \frac{\exp(2R)}{S}$$
$$\frac{1}{\nu_U} = (1-\alpha)^2 \exp(2R) \quad (21)$$

Now we can use the following procedure in order to prove Lemma 2. Given μ_G and μ_U, if (21) has a valid solution (R,S) to for some α, it gives us a lower bound on $F'(\mu_G, \mu_U)$. Specifically, we know that $E_\Lambda(R,S) = E_{sl}(R,S)$ using [6]:

$$\alpha = 1 + \frac{S}{2} - \sqrt{1 + \frac{S^2}{4}} \quad .$$

Substituting this in (21), one indeed gets the valid solution given by (19),(20). If the resulting rate is greater than R_C (18), E_Λ can be made higher. Indeed, one can verify that substituting α according to [6, (78)] still gives a valid solution (R,S); however, in that case we can give the result directly in terms of (μ_G, μ_U), since the probability to leave the Voronoi cell exactly equals that of leaving a sphere of the same volume, given by F (16) (see [9]).

The sequence of lattices Λ required in Lemma 2 is good for channel coding. Earlier we have assumed that the sequence of lattices Λ_Q is Rogers-good. These sequences satisfy the nesting relation $\Lambda \subset \Lambda_Q$. In proving the exponent of the modulo-lattice additive noise channel [6] the same conditions on the lattice sequences were used, but with the opposite nesting relation. The reversal we perform is feasible, since there exists sequences of nested lattices in which both lattices are good both in the channel-coding sense and in the Rogers sense [5].

Figure 2 depicts the different regions suggested by Lemma 2. If $1/\mu_G + 1/\mu_U > 1$, then typically the vector falls outside both the sphere and Voronoi cell, and the exponents are zero. Reducing the variances we are in the "sphere-packing region", and the probabilities to leave the sphere and the cell are exponentially equal. Reducing the variances further, the lattice error probability is larger than that of the sphere, since the setting corresponds to a channel code far from capacity. Since it is known that the straight-line exponent is not tight, it is possible to improve the combination-vector lattice exponent for some parameters, finding the corresponding lattice expurgated bound [9]. However, this is a tedious task, thus we can resort in this region to the simple bound (10).

V. DISCUSSION: THE LOSS IN THE EXPONENT

Substituting the achievable exponents $F(\cdot)$ and $F'(\cdot)$ (according to Lemma 1 and to the maximum between Lemma 2 and (10)) in (8), the lattice exponent is plotted in Figure 3, along with the optimal QG excess distortion exponent and Poltyrev exponent for comparison. It can be seen that $\underline{E}(\mu, \beta)$ converges to $E_p(\mu)$ in the high-rate limit, thus also to $F(\mu)$ for low enough μ. Under other
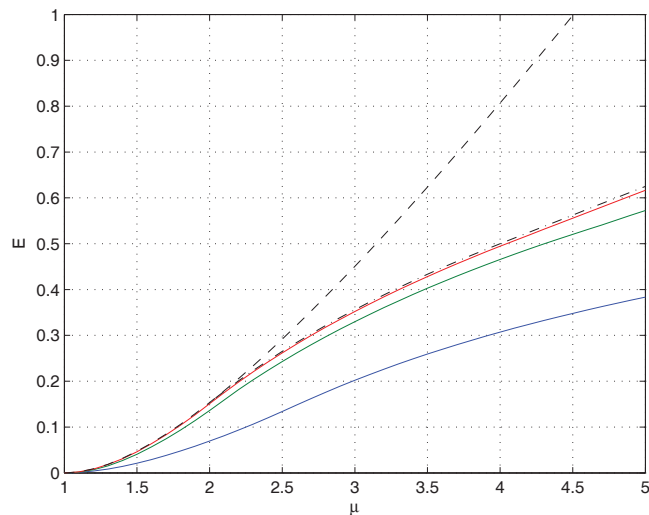
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Fig. 3: Achievable exponents as a function of the RDF SDR β and the distortion factor μ (see (5)). Solid lines are $\underline{E}(\mu, \beta)$ with β corresponding to a rate of 1, 2 and 3 nats/sample (from bottom to top). Dashed line is $F(\mu)$, while dash-dotted one is $E_p(\mu)$.

conditions, the scheme does not achieve the ideal $F(\mu)$. The loss is explained by the following two phenomena.

Granular loss. This is the gap between $E_p(\mu)$ and $\underline{E}(\mu, \beta)$. In source coding without SI, the quantizer can be made to cover the whole typical region of the source, and the exponent only reflects non-typical behavior. In our scheme, in contrast, when $\alpha_1 \alpha_2 \neq 1$ the reconstruction error contains an unbounded Gaussian element, contributing to the excess distortion events. This happens since the dithered quantizer has an additive noise model, corresponding to a forward channel realization, and then a post-factor adds bias in order to lower the distortion level. Interestingly, in [11] Zamir makes a connection between the loss in the QG WZ rate-distortion function (in a non-Gaussian setting) and additive forward channels. Indeed, this effect seems to be fundamental, and not tied to the lattice approach. A similar effect where distortion may be too high even when there is no binning error, is reported in [4] as well. In this work we show, that this loss vanishes in the limit of high rate. If the distortion were forced to be unbiased, then the forward channel realization would fit without pre/post factors, and this loss would vanish for all rates.

Shaping loss. This is the gap between $F(\mu)$ and $E_p(\mu)$. In source coding without SI, the shaping region of the quantizer may be taken to be a sphere. With SI (or when lattice quantization is desired), the shaping region must be the Voronoi cell of a lattice good for channel

coding. When the distortion constraint is far from the distortion promised by the RDF (large μ), the channel code also works at a regime far from capacity, where the probability of leaving the Voronoi cell is greater than that of leaving a sphere. Again, this does not seem to be limited to the lattice approach, as the sphere-packing bound on the AWGN error exponent is not known to be tight below the critical rate.

In light of the above, we conclude by conjecturing that for low rates and when working far from the RDF, the QG WZ excess distortion exponent is strictly smaller than the QG excess distortion exponent.

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